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**Cantor bouquets and
wandering domains for a class
of entire functions**

Yannis Dourekas

A thesis presented for the degree of
Doctor of Philosophy

The Open University
2019

Declaration

I confirm that the material contained in this thesis is the result of independent work, except where explicitly stated. None of it has previously been submitted for a degree or other qualification to this or any other university or institution.

Yannis Dourekas

Abstract

In this thesis we focus on different topological structures that arise as a result of the iteration of functions in a class of sums of exponentials, along with the different Fatou components that exist for functions in this class, making particular reference to wandering domains.

For many transcendental entire functions, the escaping set has the structure of a Cantor bouquet, consisting of uncountably many disjoint curves. Rippon and Stallard showed that there are many functions for which the escaping set has a new connected structure known as an infinite spider's web. We investigate a connection between these two topological structures for functions in our class.

The issue of whether an analytic function has wandering domains has long been of interest in complex dynamics. Sullivan proved in 1985 that rational maps do not have wandering domains. On the other hand, several transcendental entire functions have wandering domains. Using recent results on the relationship between Fatou components and the postsingular set, we prove that functions in a subset of our class do not have wandering domains. We also prove that for many of the functions the Julia set is the whole plane.

Publications

Much of the content of this thesis has been uploaded to the arXiv and submitted for publication:

- Cantor bouquets in spiders' webs, arXiv:1908.07260
- A new family of entire functions with no wandering domains, arXiv:1910.00958

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Chapter 1

Introduction

1.1 Historical overview of complex dynamics

Complex dynamics is the study of the iteration of an analytic function f . For such a function f we write

$$f^n(z) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(z)$$

for all $z \in \mathbb{C}$, where \mathbb{C} denotes the complex plane. We call f^n the n th iterate of f .

The study of complex dynamics began with Fatou [29] and Julia [34] in the early part of the 20th century. The analytic functions they initially considered were rational, and rational dynamics has a celebrated history in mathematics. For example, results on the different properties of the *Mandelbrot set*, a set that arises from the study of iteration of functions of the form $z \mapsto z^2 + c$, with $c \in \mathbb{C}$, stand out as some of the most attractive ones of mathematics in recent years [22].

This thesis is concerned with the iteration of transcendental entire functions; that is, entire functions that are not polynomials. Examples of such functions include $z \mapsto e^z$ and $z \mapsto \sin z$. The iteration of transcendental entire functions was first considered by Fatou in [30]; a comprehensive survey of relevant results in this setting was given by Bergweiler in [10].

Even though the area of complex dynamics has a history of over a century long, it was to a certain extent dormant for a significant amount of this period, although Baker, starting in the 1950s, proved many significant results in transcendental dynamics. A renaissance occurred in the 1980s, with the “no wandering domains” proof by Sullivan [62] and the pictures of the Mandelbrot set playing key roles.

Sullivan's acclaimed proof of the absence of wandering Fatou components for rational maps used innovative (for the field) techniques, such as quasiconformal surgery, and helped pave the way for further results by way of application of similar techniques by other authors. On the other hand, with the development of computing around that time, interest in complex dynamics saw an increase due to the extremely elaborate (and often self-similar) pictures that could now be produced by computers; be they either representations of the dynamic plane or the parameter plane (as, indeed, the Mandelbrot set is).

In honour of Fatou and Julia, the two most significant sets in the study of complex dynamics are named after them – and these sets partition the plane. The *Fatou set*, $F(f)$, is the set of points that exhibit “stable” behaviour, while its complement, the *Julia set*, $J(f)$, is the set of points that exhibit “chaotic” behaviour under iteration.

1.2 The Fatou and Julia sets

For the rest of the introduction, $f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic, unless specified otherwise.

Recall that a family of analytic functions \mathcal{F} on a domain $G \subset \mathbb{C}$ is called *normal* if every sequence of functions in \mathcal{F} contains a subsequence that converges locally uniformly on G to an analytic function or to infinity.

We define the Fatou set of f , $F(f)$, as the set of points $z \in \mathbb{C}$ such that $(f^n)_{n \in \mathbb{N}}$ forms a normal family in some neighbourhood of z . The Julia set of f , $J(f)$, is the complement of the $F(f)$ in \mathbb{C} . The Fatou set is open, while the Julia set is closed.

A point $z_0 \in \mathbb{C}$ is called a *periodic point* of f if $f^p(z_0) = z_0$ for some $p \in \mathbb{N}$. The smallest $p \in \mathbb{N}$ for which this holds is called the *period* of f . We say that z_0 is a *fixed point* of f in the case that $p = 1$.

We further classify periodic points according to the value of $|(f^p)'(z_0)|$. In particular, if that value is less than, equal to, or greater than 1, we say that z_0 is an *attracting*, *indifferent*, or *repelling* periodic point, respectively.

To be even more precise, if $|(f^p)'(z_0)| = 0$, we say that z_0 is *super-attracting*, and we also differentiate between cases when z_0 is indifferent. If z_0 is indifferent and $(f^p)'(z_0)$ is a root of unity, we say that z_0 is *rationally indifferent* or *parabolic*; if z_0 is indifferent and $(f^p)'(z_0)$ is not a root of unity, we say that z_0 is *irrationally indifferent*.

For a transcendental entire function attracting periodic points are in the Fatou set, while parabolic and repelling ones are in the Julia set.

We define invariance of sets as follows: a set $S \subset \mathbb{C}$ is called *forward*

invariant if $f(S) \subset S$, while S is called *backward invariant* if $f^{-1}(S) \subset S$. The set S is called *completely invariant* if it is both forward and backward invariant.

In the following theorem we will quote some of the basic properties of the Fatou and Julia sets. See [10] for statements and proofs of these results.

Theorem 1.2.1. *Let f be a transcendental entire function. Then the following hold.*

1. $F(f) = F(f^n)$ and $J(f) = J(f^n)$, for all $n \in \mathbb{N}$.
2. The Fatou and Julia sets are completely invariant.
3. Either $J(f) = \mathbb{C}$ or $\text{int}(J(f)) = \emptyset$.
4. $J(f)$ is perfect; that is, it is a closed, non-empty set with no isolated points.
5. $J(f)$ is the closure of the set of repelling periodic points of f .

There are examples of functions for which $J(f) = \mathbb{C}$; one of them is $z \mapsto e^z$ (conjectured by Fatou and proven by Misiurewicz [42]). Some more such functions will appear later in this thesis.

We now quote *Montel's theorem* (see, for example, [56, p. 54]), which is a key tool for finding points in the Fatou set.

Theorem 1.2.2 (Montel). *Let \mathcal{F} be a family of analytic functions on a domain $G \subset \mathbb{C}$. Let $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ such that, for all $z \in G$ and all $f \in \mathcal{F}$, we have $f(z) \neq z_1$ and $f(z) \neq z_2$. Then \mathcal{F} is a normal family in G .*

Let us define the orbits of points $z \in \mathbb{C}$ under f : we have the *forward orbit*,

$$O^+(z) = \{f^n(z) : n \in \mathbb{N}\},$$

and the *backward orbit*,

$$O^-(z) = \{w : f^n(w) = z \text{ for some } n \in \mathbb{N}\}.$$

We define $E(f)$ as the set of points with a finite backward orbit under f . For transcendental entire functions, $E(f)$ can contain at most one point (as follows from the great Picard theorem). The entire functions whose exceptional set does contain a point ζ , are of the form

$$z \mapsto \zeta + (z - \zeta)^m e^{g(z)},$$

for some $m \geq 0$ and some entire function g ([10, p. 156]).

An essential and striking property of the Julia set is the *blowing up property* (see, for example, [29]). Let f be an entire function, let K be a compact set such that $K \subset \mathbb{C} \setminus E(f)$ and let V be an open neighbourhood of some $z \in J(f)$. Then there exists $N \in \mathbb{N}$ such that $f^n(V) \supset K$ for all $n \geq N$.

1.3 Singular values and Fatou components

1.3.1 Singular values

In order to discuss the different kinds of components of the Fatou set (or *Fatou components*) we need some elementary knowledge of singular values. A useful reference is Sixsmith's 2018 survey on the Eremenko–Lyubich class [60].

Let $z \in \mathbb{C}$ satisfy $f'(z) = 0$. We call z a *critical point* of f . The forward image under f of a critical point z is called a *critical value* of f . We denote the sets of critical points and critical values of f as $CP(f)$ and $CV(f)$ respectively.

Let $\gamma : (0, \infty) \rightarrow \mathbb{C}$ be continuous with $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$. If $f(\gamma(t)) \rightarrow w$ as $t \rightarrow \infty$, for some finite $w \in \mathbb{C}$, we say that w is an *asymptotic value* of f . We denote the set of asymptotic values of f as $AV(f)$.

We define the set of *singular values* of f , $S(f)$, as

$$S(f) = \overline{CV(f) \cup AV(f)}.$$

To better understand the significance of the set of singular values, we can say that if $z \notin S(f)$, then there is a neighbourhood around z on which we can define every inverse branch of f .

Another set we need to consider is the *postsingular set*, $P(f)$, defined as

$$P(f) = \overline{\bigcup_{n \in \mathbb{N}} f^n(S(f))}.$$

To better understand the significance of the set of postsingular values, we can say that, if $z \notin P(f)$, then there is a neighbourhood around z on which we can define every *inverse* branch of all iterates of f .

We say that f is in the Speiser class, \mathcal{S} , if the set $S(f)$ is finite. We say that f is in the Eremenko–Lyubich class, \mathcal{B} , if $S(f)$ is bounded. Obviously, $\mathcal{S} \subset \mathcal{B}$.

These classes of functions have been studied extensively – see, for example, [24], [26] and [47].

We list two examples of functions in these classes.

- Let $f(z) := \lambda e^z$, for some $\lambda \in \mathbb{C}^*$. We have $CV(f) = \emptyset$ and $AV(f) = \{0\}$, so $f \in \mathcal{S}$.
- Let $f(z) := z^{-1} \sin z$. Then $AV(f) = \{0\}$ and there is an infinite number of critical values, all lying in the interval $[-1, 1]$. This function is in class \mathcal{B} , but not in class \mathcal{S} . See [43, p. 286].

A key tool in the study of functions in class \mathcal{B} is the idea of a *logarithmic lift*. The basic idea is to take a Jordan domain, D , so that $S(f) \cup \{0, f(0)\} \subset D$. Then, set $W := \mathbb{C} \setminus \overline{D}$, and consider the components of $V := f^{-1}(W)$, which are called *tracts* of f . Then each component of V is simply connected and bounded by a simple curve tending to infinity at both ends. The function f is a universal covering from each of these components to W . We now set $H := \exp^{-1}(W)$ and $\mathcal{T} := \exp^{-1}(V)$. We *lift* f to a map (which can be chosen to be $2\pi i$ periodic) $F : \mathcal{T} \rightarrow H$, satisfying $\exp \circ F = f \circ \exp$.

An example of an application of the logarithmic lift, is the following expansivity result by Eremenko and Lyubich [26]:

Lemma 1.3.1. *Using the notation above, we have*

$$|F'(z)| \geq \frac{1}{4\pi} (\operatorname{Re} F(z) - \log R), \text{ for } z \in \mathcal{T} \text{ such that } \operatorname{Re} F(z) > R,$$

where $R \in \mathbb{R}$ is such that $\{w \in \mathbb{C} : \operatorname{Re} w > R\} \subset H$.

1.3.2 Types of Fatou components

Let U be a component of the Fatou set of f . We call U a *Fatou component*. In this section we reference Bergweiler's survey [10, Chapter 4] frequently, as it contains a comprehensive account of results on Fatou components.

From property 2 of Theorem 1.2.1, the Fatou set is completely invariant; that is, $z \in F(f)$ if and only if $f(z) \in F(f)$. Thus, if U is a Fatou component of f , then there exist Fatou components U_n , $n \in \mathbb{N}$, so that $f^n(U) \subset U_n$. In particular, if U is bounded, then $U_n = f^n(U)$ [33].

Suppose that there exist $p > q \geq 0$ such that $U_p = U_q$. Then U is called *preperiodic*. In particular, in the case where $q = 0$, we call U *periodic*. If p is the minimum integer with the property that $U = U_p$, we say that U has period p . The set $\{U, U_1, \dots, U_{p-1}\}$ is called a *periodic cycle* of Fatou components. If U is not periodic or preperiodic, then it is called a *wandering domain*.

The behaviour of periodic Fatou components is well understood and is classified as follows (see, for example, [10]).

Theorem 1.3.2. *Let f be a meromorphic function and let U be a periodic Fatou component of f , of period p . One of the following cases holds.*

- *There exists an attracting periodic point ζ of period p in U . Then, for all $z \in U$, $f^{np}(z) \rightarrow \zeta$ as $n \rightarrow \infty$. We call U the immediate attracting basin of ζ .*
- *There exists a parabolic periodic point ζ of period p in ∂U . Then, for all $z \in U$, $f^{np}(z) \rightarrow \zeta$ as $n \rightarrow \infty$. We call U a parabolic Fatou component.*
- *There exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{D}$, so that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi ia}z$ for some $a \in \mathbb{R} \setminus \mathbb{Q}$. We call U a Siegel disk.*
- *There exists an analytic homeomorphism $\phi : U \rightarrow A$, with $A := \{z : 1 < |z| < r\}$ for some $r > 1$, so that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi ia}z$ for some $a \in \mathbb{R} \setminus \mathbb{Q}$. We call U a Herman ring.*
- *There exists $\zeta \in \partial U$ such that, as $n \rightarrow \infty$, we have $f^{np}(z) \rightarrow \zeta$, but $f^p(\zeta)$ is not defined. We call U a Baker domain.*

Note that Herman rings do not exist for entire functions, and Baker domains do not exist for rational functions.

One of the most important results in complex dynamics is Sullivan's result that wandering domains do not exist for rational functions. The proof used innovative techniques, such as quasiconformal surgery.

Wandering domains do exist for transcendental entire functions. It is known that, if U is a wandering domain for the transcendental entire function f , all limit functions of $\{f^n|_U\}$ are constant [29, Section 28]. Wandering domains for transcendental entire functions can thus be completely categorised into three groups:

- if the only limit function is ∞ , U is called *escaping*;
- if the limit functions all lie in a bounded set, U is called *of bounded orbit*; and
- if the limit functions include both finite values and ∞ , U is called *oscillating*.

Most known examples are escaping, and the first such example was constructed by Baker in 1976 [5]. Baker's construction gives an infinite product which has a bounded multiply connected Fatou component. Baker previously showed that multiply connected Fatou components exist in [2]. In fact, he proved the following in [4]:

Theorem 1.3.3. *Let f be a transcendental entire function and let U be a multiply connected Fatou component. Then U is a wandering domain and the following hold.*

- (a) *each U_n is bounded and multiply connected,*
- (b) *there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1} , for $n \geq N$, and*
- (c) *$\text{dist}(U_n, 0) \rightarrow \infty$ as $n \rightarrow \infty$.*

Another example of an escaping wandering domain was given by Herman in 1984 [32]. He showed that $z \mapsto z - 1 + e^{-z} + 2\pi i$ has an escaping wandering domain, and that this wandering domain is simply connected.

The first example of an oscillating wandering domain was given by Eremenko and Lyubich in 1987 [25], who used approximation theory, with more recent examples being given by Bishop in 2015 [16], who used quasiconformal folding, and by Martí-Pete and Shishikura in 2018 [37], who used an alternative quasiconformal approach.

Note that the existence of wandering domains where all limit functions of $\{f^n|_U\}$ lie in a bounded set is a major open question.

There are, on the other hand, several families of transcendental entire functions which have been shown not to have wandering domains, as described in [10, Section 4.6]. The methods used to find classes of functions without wandering domains usually arose as developments of Sullivan's techniques. The result was proved for class \mathcal{S} by Eremenko and Lyubich [24, 26] and Goldberg and Keen [31].

Alternative techniques to find classes of functions without wandering domains have been used, for example by Bergweiler in 1993 [12] and by Mihaljević-Brandt and Rempe-Gillen in 2013 [40].

The singular values of f are associated with Fatou components, as follows (see [10]).

Theorem 1.3.4. *Let f be meromorphic and let U be an attracting or parabolic Fatou component of period p . Then $S(f) \cap \bigcup_{m=0}^{p-1} f^m(U) \neq \emptyset$.*

Theorem 1.3.5. *Let f be meromorphic and let U be either a Siegel disk or a Herman ring. Then $\partial U \subset \overline{P(f)}$.*

We also state an immediate corollary of the above two theorems.

Corollary 1.3.6. *Let f be meromorphic and let U be a periodic Fatou component that is not a Baker domain. Then $\overline{U} \cap \overline{P(f)} \neq \emptyset$.*

In the cases of Baker domains and wandering domains, the connection is not so straightforward, but some results for these cases also exist.

For example, we know that the number of singular values needed for wandering domains to exist at all, is infinite [7, 26]. We also know that, if $f \in \mathcal{B}$, then f has neither Baker domains, nor escaping wandering domains [26, 48].

It is further known that if U is a wandering domain, then all limit functions of f^n in U lie in the derived set (that is, the set of all limit points) of $P(f)$ [13].

One other recent result is the following, shown by Barański, Fagella, Jarque and Karpińska in 2017 [9]. It describes a relationship on the distance between the postsingular set and forward images of Fatou components.

Theorem 1.3.7. *Let f be a transcendental meromorphic map and U be a Fatou component of f . Denote by U_n the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence (p_n) in $P(f)$ such that*

$$\frac{\text{dist}(p_n, U_n)}{\text{dist}(f^n(z), \partial U_n)} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

In particular, if for some $d > 0$ we have $\text{dist}(f^n(z), \partial U_n) < d$ for all n , then $\text{dist}(p_n, U_n) \rightarrow 0$ as n tends to ∞ .

1.4 The escaping and fast escaping sets

An important set in complex dynamics is the *escaping set*, $I(f)$. We define

$$I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Eremenko proved in 1989 [23] the following properties of the escaping set of a transcendental entire function f :

Theorem 1.4.1. *Let f be a transcendental entire function. Then*

1. $I(f) \cap J(f) \neq \emptyset$;
2. $J(f) = \partial I(f)$; and
3. $\overline{I(f)}$ has no bounded components.

For functions in class \mathcal{B} , Eremenko and Lyubich [26] proved the following result on the relation between the escaping set and the Julia set, using, among other results, the derivative estimate we stated in Lemma 1.3.1:

Theorem 1.4.2. *Let $f \in \mathcal{B}$. Then $J(f) = \overline{I(f)}$.*

Eremenko conjectured that $I(f)$ itself has no bounded components. This is one of the major conjectures in complex dynamics; it is referred to as *Eremenko's conjecture*, and is still open. We also quote the strong form of Eremenko's conjecture: every point in $I(f)$ can be joined to ∞ by a curve that lies in $I(f)$.

Eremenko's conjecture was motivated by many examples, including *Fatou's function*, $z \mapsto z + 1 + e^{-z}$ (see [30]), and many exponential functions for which the strong version of his conjecture holds.

Trying to answer Eremenko's conjecture has, throughout the last 30 years, led to many significant results.

Many of these were proved by considering a subset of the escaping set, the *fast escaping set*, $A(f)$, that was introduced by Bergweiler and Hinkkanen in 1999 [14]. This is the set of points that escape to infinity "as fast as possible". If we denote

$$M(r, f) = \max_{|z| \leq r} |f(z)|, \text{ for } r \geq 0,$$

and we also let $M^n(r, f)$ denote the iteration of $M(r, f)$ with respect to r , then, we can define the fast escaping set as

$$A(f) = \{z \in \mathbb{C} : \text{there exists } l \in \mathbb{N} \text{ such that } |f^{n+l}(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

with $R > 0$ being any value such that $M(r) > r$ for $r \geq R$. Note that such a value has to exist due to Montel's theorem: indeed, if $M(r) < r$, then $B(z, r) \subset F(f)$.

A thorough investigation of the fast escaping set was accomplished by Rippon and Stallard in 2012 [52]. The rest of this section refers extensively to that paper.

For each $R > 0$ with $M(r) > r$ for $r \geq R$, we can define $A_R(f)$, an important subset of $A(f)$, as:

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

This set is a helpful tool in proving different results for $A(f)$, since

$$A(f) = \cup_{l=0}^{\infty} f^{-l}(A_R(f)).$$

Rippon and Stallard proved in [52] that all the components of the fast escaping set are unbounded.

Theorem 1.4.3. *Let f be a transcendental entire function, and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then each component of $A_R(f)$ is closed and unbounded, and hence each component of $A(f)$ is unbounded.*

This was the strongest result obtained for general entire functions towards Eremenko’s conjecture, since, for any transcendental entire function, $A(f) \neq \emptyset$, and so there is at least one unbounded component of $I(f)$.

More is now known; a more recent theorem by Rippon and Stallard [54, Theorem 1.2] states the following:

Theorem 1.4.4. *Let f be a transcendental entire function. Either $I(f)$ is connected or it has infinitely many unbounded components.*

We consolidate in the following list some of the properties of the fast escaping set, found in [52].

Theorem 1.4.5. *Let f be a transcendental entire function. The following hold.*

1. $A(f) = A(f^n)$ for $n \geq 2$.
2. $A(f)$ is completely invariant.
3. $J(f) \cap A(f) \neq \emptyset$.
4. $J(f) = \partial A(f)$.
5. $A(f)$ has no bounded components.

More progress has been made on Eremenko’s conjecture for functions in class \mathcal{B} .

For example, in [57], Schleicher and Zimmer investigated the points which converge to infinity under iteration of $z \mapsto \lambda e^z$ where λ is complex and non-zero and give a complete classification of them, using symbolic dynamics; in particular, “addresses” or “itineraries”, which we also use and will define later. They showed that every escaping point of one of these functions is either on a *hair* (which we will define later as a simple curve to infinity with some specific properties), or is the *endpoint* of one. Therefore, for these functions, Eremenko’s strong conjecture holds.

In [47], Rottenfuß, Rückert, Rempe-Gillen and Schleicher, proved that Eremenko’s strong conjecture actually holds for a large subclass of \mathcal{B} . We say that the function f has *finite order* if there exist constants $c, \rho > 0$ such that, for all $z \in \mathbb{C}$, $|f(z)| \leq ce^{|z|^\rho}$.

Theorem 1.4.6. *Suppose that $f \in \mathcal{B}$ is a function of finite order, or more generally, a finite composition of functions of finite order. Then each $z \in I(f)$ can be connected to ∞ by a curve, γ , such that $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$ uniformly on γ .*

On the other hand, in the same paper they show that this is *not* the case for some functions in class \mathcal{B} :

Theorem 1.4.7. *There exists a function in class \mathcal{B} that is hyperbolic, such that each path-connected component of its Julia set $J(f)$ is bounded.*

They prove, in fact, that it is possible to construct this function so that $J(f)$ contains no non-trivial curves.

We finish this section by mentioning that, for many transcendental entire functions, the escaping set not only has no bounded components, but it has just one – connected and obviously unbounded. This structure, as introduced and defined by Rippon and Stallard [52], is referred to as a *spider's web*, and we devote an ensuing section to it.

1.5 Cantor bouquets

In this section we introduce a structure consisting of an uncountable number of pairwise disjoint curves, called a Cantor bouquet. Cantor bouquets are found in transcendental dynamics as being either inside Julia or escaping sets of functions, or the whole sets themselves.

To talk about how Cantor bouquets are usually constructed, we need some elementary concepts of symbolic dynamics.

Let $N \in \mathbb{N}$. Let Σ_N consist of all one-sided sequences $s_0s_1s_2\dots$, where each $s_j \in \mathbb{Z}$ and $|s_j| \leq N$. The one-sided shift σ on Σ_N is defined by

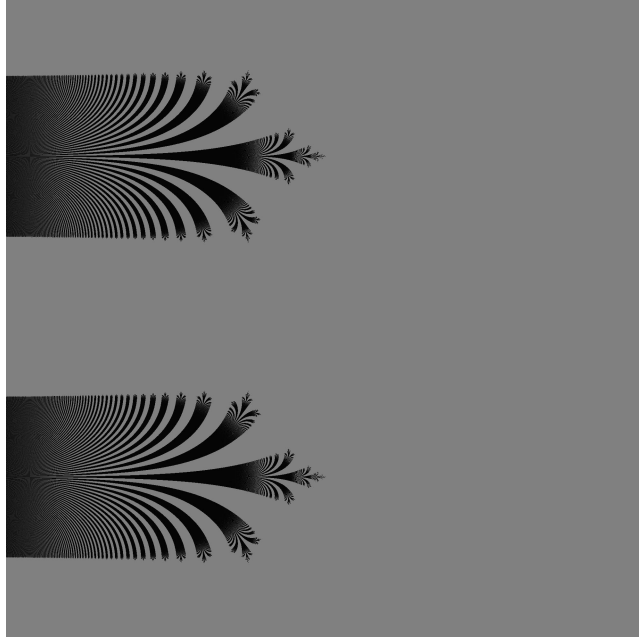
$$\sigma(s_0s_1s_2\dots) = s_1s_2s_3\dots$$

It is known that σ has dense periodic points in Σ_N , has dense orbits, and exhibits sensitive dependence on initial conditions (see for example [19, Chapter 3]).

Devaney and Tangerman [20] define a *Cantor N -bouquet* for a transcendental entire function f as follows: They call an invariant subset C of $J(f)$ an N -bouquet for f if

1. there is a homeomorphism $h : \Sigma_N \times [0, \infty) \rightarrow C$,

Figure 1.1: A Cantor bouquet: the Julia set of $z \mapsto z + 1 + e^{-z}$ as drawn by Martí-Pete; appeared in a paper by Evdoridou [27].



2. $\pi \circ h^{-1} \circ f \circ h(s, t) = \sigma(s)$, where $\pi : \Sigma_N \times [0, \infty) \rightarrow \Sigma_N$ is the projection map,
3. $\lim_{t \rightarrow \infty} h(s, t) = \infty$, and
4. $\lim_{n \rightarrow \infty} f^n \circ h(s, t) = \infty$ if $t \neq 0$.

It is easy to see that an N -bouquet is included in an $N + 1$ -bouquet, as we can just consider the sequences with absolute value less than or equal to N . Roughly speaking, an N -bouquet is a Cartesian product of a Cantor set with a half-line.

Note that $f(h(s, 0)) = h(\sigma(s), 0)$. These are called the *endpoints* of the Cantor bouquet, which we will define precisely below.

Devaney and Tangerman then define the *Cantor bouquet*, C , to be the closure of the union of all N -bouquets; that is

$$C = \overline{\cup_{n \geq 0} C_n}.$$

In the same paper, they prove that the Julia set of each of the functions $z \mapsto \lambda e^z$, where $0 < \lambda < 1/e$, is a Cantor bouquet (note that a similar result, in a more preliminary form, was proved by Devaney and Krych in [21]; the Cantor bouquet is there referred to as a “Cantor set of curves”).

Further, Devaney and Tangerman show that each of the maps $z \mapsto \lambda \sin z$, where $0 < \lambda < 1$, contains a pair of Cantor bouquets: one in which the curves tend to ∞ in the direction of the positive imaginary axis, and one in the direction of the negative imaginary axis. Moreover, they prove similar results for functions in class \mathcal{S} with hyperbolic exponential tracts.

Even though the above are consistent with the work we are doing in this thesis, we note that there is another, more topological, definition of a Cantor bouquet. The Cantor bouquets, in the sense of Devaney and Tangerman, satisfy this new definition as well.

Aarts and Oversteegen gave a complete topological description of Cantor bouquets by using *straight brushes* [1]. We quote the definition used in [8].

Definition 1.5.1. A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

- The set B is a closed subset of \mathbb{R}^2 .
- For every $(x_0, y_0) \in B$ there exists $t_{y_0} \geq 0$ such that $\{x : (x, y_0) \in B\} = [t_{y_0}, +\infty)$. The set $[t_{y_0}, +\infty) \times \{y_0\}$ is called the *hair* attached at y_0 and the point (t_{y_0}, y_0) is called its *endpoint*.
- The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n, \beta_n, \gamma_n \rightarrow y$ and $t_{\beta_n}, t_{\gamma_n} \rightarrow t_y$ as $n \rightarrow \infty$.

What Aarts and Oversteegen show in [1] is that all straight brushes are *ambiently homeomorphic* to each other. This means that there exists a homeomorphism between any two straight brushes, and, further, that homeomorphism can be chosen to be extended to a homeomorphism of \mathbb{R}^2 to itself.

We then define:

Definition 1.5.2. A *Cantor bouquet* is any subset of the plane homeomorphic to a straight brush.

Even though this topological definition is more commonly used, the N -bouquets technique is still useful. For example, Bodelón, Devaney, Hayes, Roberts, Goldberg and Hubbard [18] use it to find hairs lying in the Julia set of the functions $z \mapsto \lambda e^z$, where λ is complex and non-zero. We base many of our techniques in an upcoming chapter on their work. Further, in the same paper, they find similar hairs in the λ -plane.

Class \mathcal{B} has been linked to the Cantor bouquet structure with several results. We say that f is *hyperbolic* if the postsingular set $P(f)$ is a compact

subset of the Fatou set $F(f)$ (in particular, every function with this property is in class \mathcal{B} , since hyperbolicity implies that $S(f)$ is bounded). We further say that f is of *disjoint type* if f is hyperbolic and the Fatou set is connected. Barański, Jarque and Rempe-Gillen proved the following in [8, Theorem 1.5].

Theorem 1.5.3. *Let f be a function of disjoint type with finite order. Then the Julia set $J(f)$ is a Cantor bouquet.*

Further, they also prove the following [8, Theorem 1.6].

Theorem 1.5.4. *Let $f \in \mathcal{B}$ be of finite order. Then there exists a forward invariant Cantor bouquet X in the Julia set $J(f)$.*

One of the most striking properties of some Cantor bouquets concerns the Hausdorff dimension of their endpoints. In particular, results were proven for $E_\lambda(z) := \lambda e^z$ where λ is a complex non-zero variable. We denote by $\dim S$ the Hausdorff dimension of $S \subset \mathbb{C}$. McMullen, in 1987 [38], proved the following:

Theorem 1.5.5. *Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\dim J(E_\lambda) = 2$.*

Karpińska, then, in 1999 [36] proved the following result, which is referred to as “Karpińska’s paradox” due to its counterintuitive nature.

Theorem 1.5.6. *Let $0 < \lambda < 1/e$ and let C_λ be the set of endpoints of the Cantor bouquet that forms $J(E_\lambda)$. Then $\dim(J(E_\lambda) \setminus C_\lambda) = 1$.*

Note that it follows from Theorems 1.5.5 and 1.5.6 that $\dim C_\lambda = 2$. This also had been previously proven by Karpińska [35].

Roughly speaking, this result shows how “dense” the endpoints are in the Cantor bouquet – we have new curves originating throughout.

Bergweiler, in 2010 [11], proved a three-dimensional analogue of Karpińska’s paradox, for Zorich maps, which are quasiregular self-maps of \mathbb{R}^3 , and are analogous to the exponential maps in \mathbb{C} .

1.6 Spiders’ webs

For many transcendental entire functions, the escaping, fast escaping, or Julia set has a connected structure known as a spider’s web. Defined and studied by Rippon and Stallard in [52], which we refer to frequently in this section, the definition of a spider’s web is as follows.

Definition 1.6.1. A set E is an *(infinite) spider’s web* if E is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$, for $n \in \mathbb{N}$, $\partial G_n \subset E$, for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}$.

This structure provides a stark contrast with that of the uncountable number of hairs of the Cantor bouquet. Nevertheless, in this thesis we prove that we can find Cantor bouquets inside spiders' webs.

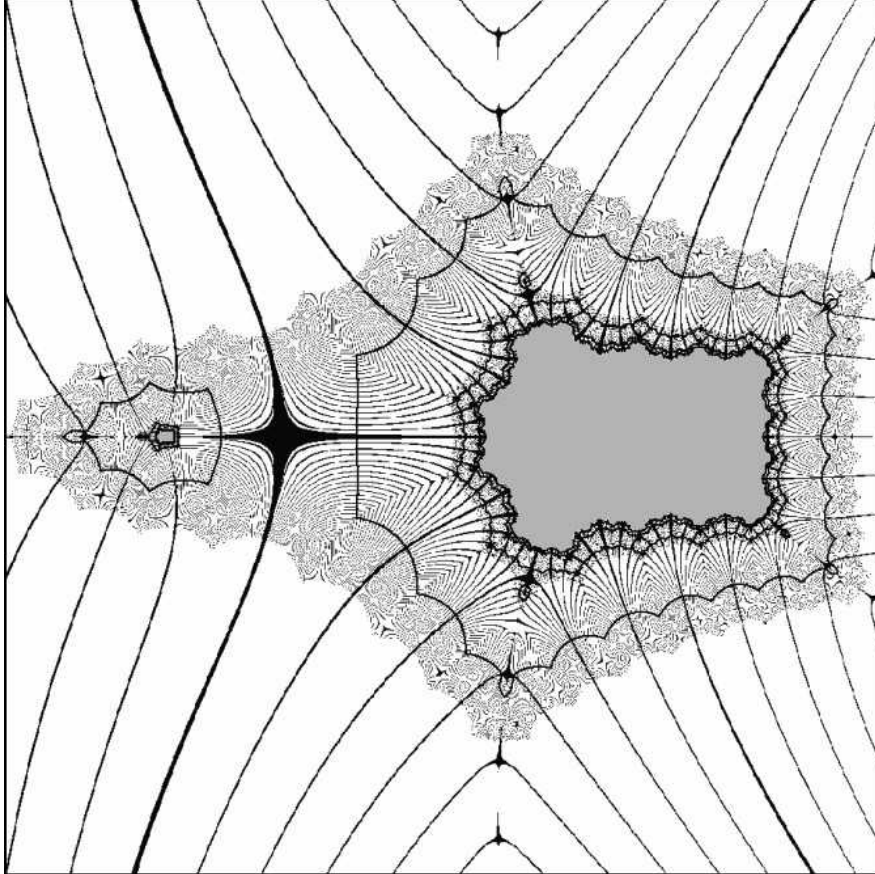


Figure 1.2: An approximation of an $A_R(f)$ spider's web, created by Dominique Fleishmann, for the function $z \mapsto (1/2)(\cos z^{1/4} + \cosh z^{1/4})$. The greyed out region in the right-hand side is an attracting Fatou basin. The black curves all belong in the spider's web.

We state some results about spider's webs. The following result was proven by Rippon and Stallard [52, Theorem 1.4].

Theorem 1.6.2. *Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. If $A_R(f)^c$ has a bounded component, then each of $A_R(f)$, $A(f)$ and $I(f)$ is a spider's web.*

In particular, it follows from Theorem 1.6.2 that if $A_R(f)$ is a spider's web, then so are $A(f)$ and $I(f)$.

There are many functions for which $A_R(f)$ is a spider's web. For all these functions, the sets $A_R(f)$, $A(f)$ and $I(f)$ are connected, so Eremenko's conjecture holds. The following result was proven in [52, Theorem 1.5].

Theorem 1.6.3. *Let f be a transcendental entire function, let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$, and let $A_R(f)$ be a spider's web.*

1. *If f has no multiply connected Fatou components, then each of*

$$A_R(f) \cap J(f), A(f) \cap J(f), I(f) \cap J(f), \text{ and } J(f)$$

is a spider's web.

2. *The function f has no unbounded Fatou components.*

Baker, in 1981 [6], asked whether small growth of f implies the boundedness of every component of $F(f)$; in particular, the growth condition would be for f to be of at most order $1/2$, minimal type. This is known as *Baker's conjecture*.

We quote [52, Theorem 1.9]; a result about classes of functions with $A_R(f)$ spiders' webs. Items (b) and (c) below, together with Theorem 1.6.3, give a partial answer to Baker's conjecture.

Theorem 1.6.4. *Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if one of the following holds:*

- (a) *f has a multiply connected Fatou component;*
- (b) *f has very small growth; that is, there exist $m \geq 2$ and $r_0 > 0$ such that*

$$\log \log M(r, f) < \frac{\log r}{\log^m r}, \text{ for } r > r_0,$$

where \log^m is the m th iterated logarithm;

- (c) *f has order less than $1/2$ and regular growth;*
- (d) *f has finite order, Fabry gaps and regular growth;*
- (e) *f has a sufficiently strong version of the pits effect and has regular growth.*

The proof for the cases (a), (b) and (c) follows from results [50, 51]. For (d) and (e), see [52].

We say that the transcendental entire function f has Fabry gaps if there are significant gaps in its power series representation. To be precise, we say that f has Fabry gaps if

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k},$$

with $n_k/k \rightarrow \infty$ as $k \rightarrow \infty$.

We say that f exhibits the pits effect, if, roughly, f is very large except in small neighbourhoods around its zeros (the so-called pits).

Mihaljević-Brandt and Peter [39] considered Poincaré functions of polynomials and proved that, under certain conditions, the (fast) escaping sets of these functions are spiders' webs.

Evdoridou proved in [27] that the escaping set of Fatou's function, $z \mapsto z + 1 + e^{-z}$, is a spider's web. In fact, this is an example of a function whose escaping set is a spider's web, but its fast escaping set is not. In the same paper, she further showed that this result implies that infinity, together with the non-escaping endpoints of the Julia set of Fatou's function, forms a totally disconnected set.

For more examples of classes of functions whose escaping and fast escaping sets are spiders' webs, see, for example, [58] by Sixsmith.

Julia and escaping set spiders' webs also appear for transcendental self-maps of the punctured complex plane, as proven in 2019 by Evdoridou, Martí-Pete and Sixsmith [28]. In the same paper, they conjecture that there is no transcendental self-map of the punctured plane for which the fast escaping set is a spider's web.

The following result [52, Theorem 1.6] gives insight into the extremely elaborate nature of spiders' webs.

Theorem 1.6.5. *Let f be a transcendental entire function, let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $A_R(f)$ be a spider's web. Then*

1. *All the components of $A(f)^c$ are compact.*
2. *Each point in $J(f)$ is the limit of a sequence of points, each of which lies in a distinct component of $A(f)^c$.*

Further, and related to the second property of the above theorem, Osborne proved in [44] that singleton components of $A(f)^c$ are dense in $J(f)$. In the same paper he also proves the following result [44, Theorem 1.5 (a)(ii)], which we will make use of in Chapter 4.

Theorem 1.6.6. *Let f be a transcendental entire function, let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$, and let $A_R(f)$ be a spider's web. Suppose that K is a component of $A(f)^c$ with bounded orbit. Then, if the interior of K is non-empty, this interior consists of non-wandering Fatou components.*

An immediate corollary of this theorem is the following:

Corollary 1.6.7. *Let f be a transcendental entire function, let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$, and let $A_R(f)$ be a spider's web. Then f has no wandering domains with bounded orbit.*

As mentioned above, the question of whether $I(f)$ can be a spider's web when $A(f)$ is not was answered positively by Rippon and Stallard in [53], while Evdoridou also has an example in [27]. We can state the following open question from [52] about spider's webs: Can $A(f)$ be a spider's web when $A_R(f)$ is not?

Note that, if $A_R(f)$ is a spider's web, where $R > 0$ is such that $M(r, f) > r$ for $r \geq R$, then f does not belong to class \mathcal{B} , since there is no path to infinity on which f is bounded [52, Theorem 1.8].

1.7 Objectives and structure of this thesis

In this thesis we study a specific family of transcendental entire functions; a sum of rotations of the exponential with different coefficients. Sixsmith proved that the Julia set of each function in this class is a spider's web [59]. We show that, for certain coefficients, in these spiders' webs there reside Cantor bouquets. We then use a new technique to show that a certain subfamily does not have wandering domains and, further, the Julia set is the whole complex plane for many of these functions.

The structure of the thesis is as follows.

- In Chapter 2 we fix some notation, introduce the family we are studying, quote some its properties and prove some new ones.
- In Chapter 3 we prove the existence of Cantor bouquets for functions in this family with certain coefficients.
- In Chapter 4 we prove the absence of wandering domains for a subfamily, as well as the fact that the Julia set is the whole complex plane for functions with certain coefficients in this subfamily.
- In Chapter 5 we discuss questions and plans for future work.

Chapter 2

Preliminaries

Sixsmith studied in [59] the family of transcendental entire functions, defined for $p \in \mathbb{N}$ with $p \geq 3$ by

$$\mathcal{E}_p = \left\{ f : f(z) = \sum_{k=0}^{p-1} a_k \exp(\omega_p^k z), \text{ with } a_k \in \mathbb{C}^* \text{ and } k \in \{0, 1, \dots, p-1\} \right\},$$

where $\omega_p = \exp(2\pi i/p)$ is a p th root of unity. Note that for $p = 2$ we have the cosine family

$$\mathcal{E}_2 = \{f : f(z) = a_0 e^z + a_1 e^{-z}, \text{ with } a_0, a_1 \in \mathbb{C}^*\},$$

and for $p = 1$ we have the exponential family

$$\mathcal{E}_1 = \{f : f(z) = \lambda e^z, \text{ with } \lambda \in \mathbb{C}^*\},$$

which was mentioned frequently in our introduction.

Some of the results in this thesis were proven for \mathcal{E}_1 and \mathcal{E}_2 . Here we consider the families \mathcal{E}_p , for $p \geq 3$, where new techniques are needed.

For each $p \geq 3$ we also define the subfamily of \mathcal{E}_p ,

$$\mathcal{F}_p = \left\{ f_\lambda : f_\lambda(z) = \lambda \sum_{k=0}^{p-1} \exp(\omega_p^k z), \text{ for some } \lambda \in \mathbb{R}^* \right\}, \quad (2.0.1)$$

where $\omega_p = \exp(2\pi i/p)$ is a p th root of unity.

The family \mathcal{F}_p provides us with strong symmetry properties (that we make explicit below) and for that reason, among others, many of our results concern it, rather than \mathcal{E}_p . Obviously, all results that hold for \mathcal{E}_p hold for \mathcal{F}_p as well.

In this chapter we quote some of Sixsmith's results for \mathcal{E}_p with $p \geq 3$ from [59], and also prove some new ones.

The key property of \mathcal{E}_p is that, for each $p \geq 3$, there exist p unbounded regions outside a circle centered at the origin with the property that f behaves like a single exponential in each one of them. Each of these regions is a $2k\pi/p$ rotation of the others, for $k = 1, \dots, p-1$ (see Figure 2.1).

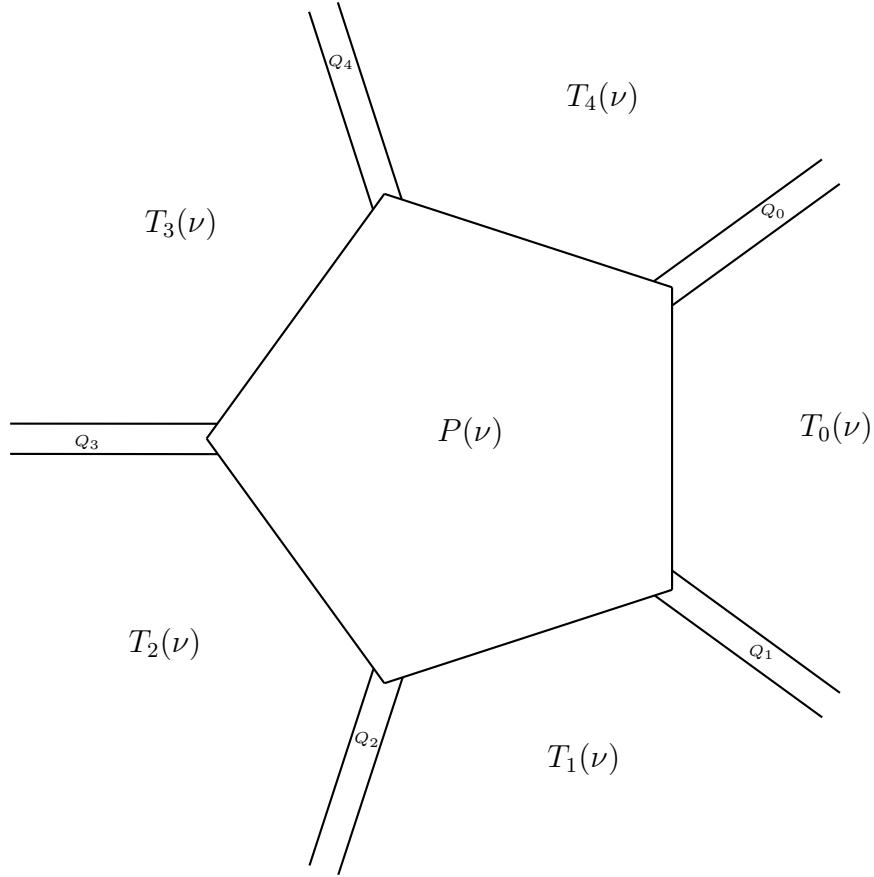


Figure 2.1: The sets $P(\nu)$, $T_k(\nu)$ and Q_k for $p = 5$ and $k \in \{0, 1, 2, 3, 4\}$.

We now make this partition explicit. Choose a constant σ such that

$$0 < \sigma < \frac{1}{8\sqrt{2}}.$$

Fix a constant $\eta > 4/\sigma$. Fix also a constant τ sufficiently large that

$$\tau \geq \frac{1}{2 \sin(\pi/p)} \log \frac{4p\eta \max\{|a_k| : 0 \leq k \leq p-1\}}{\min\{|a_k| : 0 \leq k \leq p-1\}} > 0.$$

Note here that if we are working with \mathcal{F}_p , the condition on the constant τ becomes

$$\tau \geq \frac{\log(4p\eta)}{2\sin(\pi/p)}.$$

Suppose that $\nu > 0$ is large compared to τ ; we will specify its size more precisely below. Let $P(\nu)$ be the interior of the regular p -gon that is centered at the origin and has vertices at the points

$$\frac{\nu}{\cos(\pi/p)} \exp\left(\frac{(2k+1)i\pi}{p}\right), \text{ for } k \in \{0, 1, \dots, p-1\}.$$

Define the domains

$$Q_k = \left\{ z \exp\left(\frac{(-2k+1)i\pi}{p}\right) : \operatorname{Re}(z) > 0, |\operatorname{Im}(z)| < \tau \right\}, \quad (2.0.2)$$

for $k \in \{0, 1, \dots, p-1\}$. Roughly speaking, each Q_k can be obtained by rotating a half-infinite horizontal strip of width 2τ around the origin until a vertex of $P(\nu)$ is positioned centrally in the strip.

Set

$$T(\nu) = \mathbb{C} \setminus \left(P(\nu) \cup \bigcup_{k=0}^{p-1} Q_k \right).$$

The set $T(\nu)$ consists of p simply connected unbounded components. These components are arranged rotationally symmetrically. We label these $T_j(\nu)$, for $j \in \{0, 1, \dots, p-1\}$, where $T_0(\nu)$ is the component that has an unbounded intersection with the positive real axis. Then, each component of $T_{j+1}(\nu)$ is obtained by rotating $T_j(\nu)$ clockwise around the origin by $2\pi/p$ radians; see Figure 1. We take $\nu > 0$ so large that

$$|e^z| \geq 4p\eta |\exp(\omega_p^k z)| \quad (2.0.3)$$

for $k = 1, 2, \dots, p-1$ and $z \in T_0(\nu)$.

The following lemma concerns the behaviour of f in each component of $T(\nu)$. It is a modified version of [59, Lemma 4.1].

Lemma 2.0.1. *Let $f \in \mathcal{E}_p$, $p \geq 3$. Suppose that $\eta, \tau, T_j(\nu)$ and $T(\nu)$ are as defined above, for $j \in \{0, 1, \dots, p-1\}$. Then there exists $\nu' > 0$ such that the following holds. Suppose that $\nu \geq \nu'$; there exists a constant $\varepsilon_0 \in (0, 1)$, independent of ν , such that, for all $z \in T(\nu)$,*

$$|f'(z)| > 2 \quad (2.0.4)$$

$$\left| z \frac{f'(z)}{f(z)} \right| > 2, \quad (2.0.5)$$

and finally

$$|f(z)| > \max\{e^{\varepsilon_0 \nu}, M(\varepsilon_0 |z|, f)\}. \quad (2.0.6)$$

McMullen proved that, even though for many functions $f \in \mathcal{E}_1$ the area of $J(f)$ is zero, if $f \in \mathcal{E}_2$, then f has positive area [38]. Sixsmith generalised this result as follows [59, Theorem 1.1].

Theorem 2.0.2. *Let $f \in \mathcal{E}_p$, $p \geq 2$. Then $J(f) \cap A(f)$ has positive area.*

More importantly for us, he also proved the following.

Theorem 2.0.3. *Suppose that $f \in \mathcal{E}_p$, where $p \geq 3$. Then each of*

$$A_R(f), A(f), I(f), J(f) \cap A_R(f), J(f) \cap A(f), J(f) \cap I(f), \text{ and } J(f)$$

is a spider's web.

We prove the following lemma concerning points that stay in $T_0(\nu)$ under iteration.

Lemma 2.0.4. *Let $z \in \mathbb{C}$ be such that $f^n(z) \in T(\nu)$ for all $n \geq 1$. If ν is large enough, then $z \in J(f) \cap A(f)$.*

To prove this we need the following results: First, we need a lemma from [59].

Lemma 2.0.5. *Suppose that f is a transcendental entire function and that $z_0 \in I(f)$. Set $z_n = f^n(z_0)$, for $n \in \mathbb{N}$. Suppose that there exist $\lambda > 1$ and $N \geq 0$ such that*

$$f(z_n) \neq 0 \quad \text{and} \quad \left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \lambda, \quad \text{for } n \geq N.$$

Then either z_0 is in a multiply connected Fatou component of f or $z_0 \in J(f)$.

Second, we state a corollary from [59] (proved using [12, Theorem 4.5]).

Corollary 2.0.6. *Suppose $f \in \mathcal{E}_p$, $p \geq 1$. Then f has no multiply connected Fatou components.*

Proof of Lemma 2.0.4. Let $z \in \mathbb{C}$ be such that $f^n(z) \in T_j(\nu)$ for all $n \geq 0$ and $j \in \{0, 1, \dots, p-1\}$. If ν is large enough, it follows from (2.0.6) that $z \in I(f)$. We will prove that $z \in A(f)$. Let $0 < \varepsilon_0 < 1$ be the constant from Lemma 2.0.1. We need a standard result about $M(r)$ (see, for example, [52, p. 7, (2.3)]): if $k > 1$, then

$$\frac{M(kr)}{M(r)} \rightarrow \infty \text{ as } r \rightarrow \infty.$$

Therefore, taking $k = 1/\varepsilon_0$, we have

$$M(r) = M\left(\frac{\varepsilon_0 r}{\varepsilon_0}\right) \geq \frac{1}{\varepsilon_0^2} M(\varepsilon_0 r) \quad (2.0.7)$$

for all large enough $r > 0$, say $r \geq \varepsilon_0 r_0 > 0$. Further, from (2.0.6), we have

$$|f(z)| \geq M(\varepsilon_0 |z|)$$

since $z \in T_j(\nu)$ for some $j \in \{0, 1, \dots, p-1\}$, and thus, by (2.0.7) with $|z| \geq \varepsilon_0 r_0$,

$$|f(z)| \geq \frac{1}{\varepsilon_0^2} M(\varepsilon_0^2 |z|).$$

Additionally, by substitution and the previous inequality,

$$|f^2(z)| \geq \frac{1}{\varepsilon_0^2} M(\varepsilon_0^2 |f(z)|) \geq \frac{1}{\varepsilon_0^2} M(M(\varepsilon_0^2 |z|)),$$

and thus, using induction, we have

$$|f^n(z)| \geq \frac{1}{\varepsilon_0^2} M^n(\varepsilon_0^2 |z|) \geq M^n(\varepsilon_0^2 |z|)$$

for all $n \geq 0$ and all large enough $|z|$. Therefore, for these z with large enough moduli, $z \in A(f)$. But the other $z \in \mathbb{C}$ for which $f^n(z) \in T_j(\nu)$ for all $n \geq 0$ and $j \in \{0, 1, \dots, p-1\}$ are in $I(f)$, so their moduli will get as large as we want, making them preimages of points in $A(f)$. Consequently, they are in $A(f)$ as well.

We now prove that $z \in J(f)$. From (2.0.5) we have

$$\left| f^n(z) \frac{f'(f^n(z))}{f(f^n(z))} \right| > 2$$

for $n \geq 0$. Since $z \in I(f)$, it follows from Lemma 2.0.5 that either $z \in J(f)$ or z is in a multiply connected Fatou component. The latter case is impossible by Corollary 2.0.6, so $z \in J(f)$. \square

We talked about the symmetry of \mathcal{F}_p , $p \geq 3$. Below is a specific result that will allow us to work in certain angles in order to prove results for the whole plane, and justifies the use of the phrase “due to symmetry” that will appear numerous times in what follows.

Lemma 2.0.7. *Let $f \in \mathcal{F}_p$, $p \geq 3$, and let $\lambda = 1$. Then $f(\omega_p^k z) = f(z)$ for $k = 1, \dots, p-1$ and for all $z \in \mathbb{C}$.*

Proof. It suffices to prove that $f(\omega_p z) = f(z)$. We have

$$f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z),$$

so

$$\begin{aligned} f(\omega_p z) &= \sum_{k=0}^{p-1} \exp(\omega_p^k \omega_p z) = \sum_{k=0}^{p-1} \exp(\omega_p^{k+1} z) \\ &= \sum_{k=1}^{p-1} \exp(\omega_p^k z) + \exp(\omega_p^p z) = \sum_{k=1}^{p-1} \exp(\omega_p^k z) + \exp(z) \\ &= \sum_{k=0}^{p-1} \exp(\omega_p^k z) = f(z). \end{aligned} \quad \square$$

Finally, we briefly describe our use of symbolic dynamics. We fix the notation found in [18] that we also use in this thesis. For each integer k , we define horizontal strips $R(k)$ by

$$R(k) = \{z \in \mathbb{C} : (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi\}.$$

Note that $z \mapsto e^z$ maps the boundary of $R(k)$ onto the negative real axis and $z \mapsto e^z$ maps $R(k)$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ for each integer k .

Definition 2.0.8. For $z \in \mathbb{C}$, the *itinerary* of z under f is the sequence of integers $s(z) = s_0 s_1 s_2 \dots$ where $s_n = k$ if and only if $f^n(z) \in R(k)$. We do not define the itinerary of z if $f^n(z) \in \cup_{k \in \mathbb{N}} \partial R(k)$ for some n .

Chapter 3

Cantor bouquets in spiders' webs

3.1 Statement of main result

In the introductory chapter of this thesis, we introduced and described two distinct topological structures, Cantor bouquets (Section 1.5) and spiders' webs (Section 1.6). These structures arise as Julia or escaping sets of transcendental entire functions.

On the surface, these structures appear to be very different; a Cantor bouquet consists of an uncountable number of pairwise disjoint curves, while a spider's web is a connected structure comprising a sequence of holes and loops. Nevertheless, in this chapter we prove that for some transcendental entire functions, it is possible to find Cantor bouquets inside a Julia set spider's web.

Recall that Theorem 2.0.3 states that the Julia, escaping and fast escaping sets of each of the functions in the family \mathcal{F}_p , $p \geq 3$, is a spider's web. We prove that, for a smaller family, within each of these spiders' webs there lie Cantor bouquets. We formulate our main result as follows:

Theorem 3.1.1. *Let $f_\lambda \in \mathcal{F}_p$, $p \geq 3$, and let $\lambda = 1$. Then $J(f)$ is a spider's web that contains a Cantor bouquet. Additionally, the curves minus the endpoints lie in $A(f)$.*

For simplicity, we write

$$\mathcal{F} := \left\{ f : f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z) \text{ for some } p \geq 3 \right\},$$

where $\omega_p = \exp(2\pi i/p)$ is a p th root of unity.

Remark. The result of Theorem 3.1.1 holds for functions in \mathcal{F}_p , $p \geq 3$, where $\lambda > 0$. The reasoning is similar, but for simplicity we only give the proof for the case $\lambda = 1$. Additionally, when p is even, $f \in \mathcal{F}_p$, $p \geq 3$, is an even function, and thus the result holds for negative λ as well.

Recall that one of the main properties we mentioned in Chapter 2 is that, for each $p \geq 3$, for $f \in \mathcal{F}_p$ there exist p unbounded regions outside a p -gon centered at the origin where we have good control over the behaviour of f . Each of these regions is a $2k\pi/p$ rotation of the others, for $k = 1, \dots, p-1$ (see Lemma 2.0.1).

We will prove our result for just one of the p regions mentioned above; due to symmetry, the curves we find in that region will have $2k\pi/p$ rotations in all the other regions for $k = 1, \dots, p-1$, and these rotations will have similar properties.

Throughout the rest of this chapter p is a fixed integer with $p \geq 3$ and $f \in \mathcal{F}$.

The structure of the proof is as follows:

- In Section 3.2 we find all the zeros and critical points for all $f \in \mathcal{F}$.
- In Section 3.3 we find different subsets of the plane that cover themselves under f : the endpoints of the curves in the Cantor bouquet will lie in these regions.
- In Section 3.4 we identify curves that extend to infinity (which we have defined as *hairs*), prove that they do, in fact, constitute a Cantor bouquet, and further prove that they are in $J(f)$ and $A(f)$ (apart from the endpoints), thus making them part of the $J(f)$, $A(f)$, and $A(f) \cap J(f)$ spiders' webs.

Our argument in Sections 3.3 and 3.4 is inspired by [18], where the authors prove the existence of hairs in the dynamical plane for the family of complex exponential functions $z \mapsto \lambda e^z$ for $\lambda \in \mathbb{C}$. In our case, the functions in \mathcal{F} provide extra challenges (for example, locating the critical points and finding regions that cover themselves under iteration), since they arise as sums of exponentials and, further, are not in the Eremenko–Lyubich class. Several different tools, including Laguerre's theorem, are thus required in order to prove our results.

3.2 Zeros and critical points

Recall that $f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z)$ for some fixed $p \geq 3$. In this section we will locate the zeros of f . These will, in turn, lead us to the location of the

critical points and the critical values of f , and later on allow us to locate bounded sets that cover themselves under iteration.

We claim that all the zeros of f lie on the rays V_0, \dots, V_{p-1} , where $V_0 := \{x + iy \in \mathbb{C} : y = \tan(\pi/p)x, x > 0\}$ and V_1, \dots, V_{p-1} are its $2k\pi/p$ -rotations around the origin for $k = 1, \dots, p-1$ respectively. The main tool used to prove this is the following result which we state as a lemma [46, Problem 160]:

Lemma 3.2.1. *Let q be an integer, $q \geq 2$. The entire function*

$$F(z) = 1 + \frac{z}{q!} + \frac{z^2}{(2q)!} + \frac{z^3}{(3q)!} + \dots$$

has no non-real zeros.

We now state and prove our result about the zeros of f .

Theorem 3.2.2. *Let $f \in \mathcal{F}$. Then all the zeros of f lie on the rays V_0, \dots, V_{p-1} .*

Proof. We write

$$f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z) = \sum_{k=0}^{p-1} \sum_{j=0}^{\infty} \left(\frac{(\omega_p^k z)^j}{j!} \right) = p \left(1 + \frac{z^p}{p!} + \frac{z^{2p}}{(2p)!} + \dots \right),$$

since $1 + \omega_p^k + (\omega_p^k)^2 + \dots + (\omega_p^k)^{p-1} = 0$ for $k = 1, \dots, p-1$.

We substitute $z = w^{1/p}$, the principal branch, to obtain

$$g(w) = f(w^{1/p}) = p \left(1 + \frac{w}{p!} + \frac{w^2}{(2p)!} + \dots \right). \quad (3.2.1)$$

We can apply Lemma 3.2.1 to the function g to deduce that all the zeros of g are real. Hence, the zeros of f must lie on the preimages of the real axis under $z \mapsto z^{1/p}$. These are exactly the rays V_0, \dots, V_{p-1} , which are the preimages of the negative real axis, along with the positive real axis, which is the preimage of itself. But it is simple to see that $f(x) \neq 0$ for $x \geq 0$. \square

In fact, we can say more about the location of the zeros. In particular, we can describe their distribution in a neighbourhood of infinity.

First, we introduce some notation and establish some symmetry properties of f . Consider one of the terms of the sum defining f ; its general form is

$$\exp(\omega_p^k z) = \exp(e^{2ik\pi/p}(x + iy)) = \exp(u_k(z)) \exp(iv_k(z)),$$

with $z = x + iy$,

$$u_k(z) = x \cos\left(\frac{2k\pi}{p}\right) - y \sin\left(\frac{2k\pi}{p}\right) \quad (3.2.2)$$

and

$$v_k(z) = x \sin\left(\frac{2k\pi}{p}\right) + y \cos\left(\frac{2k\pi}{p}\right). \quad (3.2.3)$$

We prove our results for V_0 ; analogous results follow for the rest of the rays due to symmetry. We start by proving a lemma that simplifies the equation of f on V_0 , in particular showing that f is real on V_0 , and so f is real on each of the rays V_k , $k \in \{1, \dots, p-1\}$, by symmetry; this fact also follows from (3.2.1), since if $z \in V_k$, then $z^p \in \mathbb{R}$ and $f(z) = g(z^p)$.

Lemma 3.2.3. *Let $z \in V_0$. We have*

$$f(z) = 2 \sum_{k=0}^{p/2-1} \exp(u_k(z)) \cos(v_k(z))$$

for even p , and

$$f(z) = 2 \sum_{k=0}^{(p-1)/2} \exp(u_k(z)) \cos(v_k(z))$$

for odd p .

Proof. We show that the terms of the sum defining f act similarly in pairs on the ray V_0 with regards to their moduli, which are proven to be equal for specific pairs, as well as their arguments which, for the same pairs, are proven to be of opposite sign. In particular, the k -th term, for $k < (p-1)/2$, behaves similarly to the $(p-k-1)$ -th term.

In particular, a point in V_0 is of the form $r \exp(\pi i/p)$ for $r > 0$. By substitution we get

$$u_k(r \exp(\pi i/p)) = \operatorname{Re} \exp(e^{2\pi i k/p} r e^{\pi i/p}) = e^r \cos(\pi(2k+1)/p)$$

and

$$v_k(r \exp(\pi i/p)) = \operatorname{Im} \exp(e^{2\pi i k/p} r e^{\pi i/p}) = e^r \sin(\pi(2k+1)/p).$$

On the other hand,

$$\begin{aligned} u_{p-k-1}(r \exp(\pi i/p)) &= \operatorname{Re} \exp(e^{2\pi i(p-k-1)/p} r e^{\pi i/p}) \\ &= e^r \cos(\pi(2p-2k-1)/p) \\ &= e^r \cos(\pi(-2k-1)/p), \end{aligned}$$

so

$$u_k(r \exp(\pi i/p)) = u_{p-k-1}(r \exp(\pi i/p)), \quad (3.2.4)$$

and

$$\begin{aligned} v_k(r \exp(\pi i/p)) &= \operatorname{Im} \exp(e^{2\pi i k/p} r e^{\pi i/p}) = e^r \sin(\pi(2p - 2k - 1)/p) \\ &= e^r \sin(\pi(-2k - 1)/p), \end{aligned}$$

so

$$v_k(r \exp(\pi i/p)) = -v_{p-k-1}(r \exp(\pi i/p)). \quad (3.2.5)$$

Therefore, for $z = r \exp(\pi i/p)$, the following sum is real:

$$\begin{aligned} \exp(\omega_p^k z) + \exp(\omega_p^{p-k-1} z) &= \exp(u_k(z)) \exp(iv_k(z)) + \exp(u_{p-k-1}(z)) \exp(iv_{p-k-1}(z)) \\ &= \exp(u_k(z)) \exp(iv_k(z)) + \exp(u_k(z)) \exp(-iv_k(z)) \\ &= 2 \exp(u_k(z)) \cos(v_k(z)). \end{aligned}$$

Note that, if p is odd, say $p = 2m + 1$, then, by (3.2.5),

$$v_m(r \exp(\pi i/p)) = v_{(2m+1)-m-1}(r \exp(\pi i/p)) = -v_m(r \exp(\pi i/p))$$

so

$$v_m(r \exp(\pi i/p)) = 0$$

and thus the only term of the sum that does not belong to a pair, exclusively attains real values on V_0 . Further, in this case

$$u_m(r \exp(\pi i/p)) = r \cos\left(\frac{\pi(2m+1)}{2m+1}\right) = -r,$$

so $\exp(u_m(r \exp(\pi i/p))) = e^{-r}$. For points $z \in V_0$, we can therefore write

$$f(z) = 2 \sum_{k=0}^{p/2-1} \exp(u_k(z)) \cos(v_k(z))$$

for even p , and

$$f(z) = 2 \sum_{k=0}^{(p-1)/2} \exp(u_k(z)) \cos(v_k(z))$$

for odd p . □

We state an elementary lemma about real exponentials which we will make use of below several times.

Lemma 3.2.4. *For $d > 0$ and $a < 1$, let $E_{d,a} : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $E_{d,a}(x) = e^x - de^{ax}$. Then, for $x(1-a) > \log^+(ad)$, and as $x \rightarrow \infty$, $E_{d,a}(x)$ increases to infinity.*

Proof. We have

$$E_{d,a}(x) = e^x - de^{ax} = e^{ax}(e^{x(1-a)} - d) \rightarrow \infty$$

as $x \rightarrow \infty$, and is increasing for $x(1-a) > \log^+(ad)$. \square

We are following the notation and the partition of the plane as introduced in [59, p. 9757] and discussed in Chapter 2 (see Lemma 2.0.1 and Figure 2.1). From Lemma 2.0.1 we know that there exist no zeros of f in $T_0(\nu)$ and all potential zeros of f should therefore lie in $\cup_{k=0,\dots,p-1} Q_k$. Due to symmetry, it suffices to locate all zeros in Q_0 ; the rest will be $2\pi/p$ rotations of these. We can write

$$Q_0 = \{z \in \mathbb{C} : z = w + t \text{ for } w \in V_0 \text{ and } |t| \leq \tau / \cos(\pi/p)\}.$$

We define the lines

$$C_m = \left\{ x + iy : y = -\cot(\pi/p)x + \frac{m\pi}{\sin^2(\pi/p)} \right\} \quad (3.2.6)$$

for $m \in \mathbb{N}$. Note that C_m meets V_0 at $(m\pi \cot(\pi/p), m\pi)$. As pointed out in [59, p. 9766], it is easy to check that

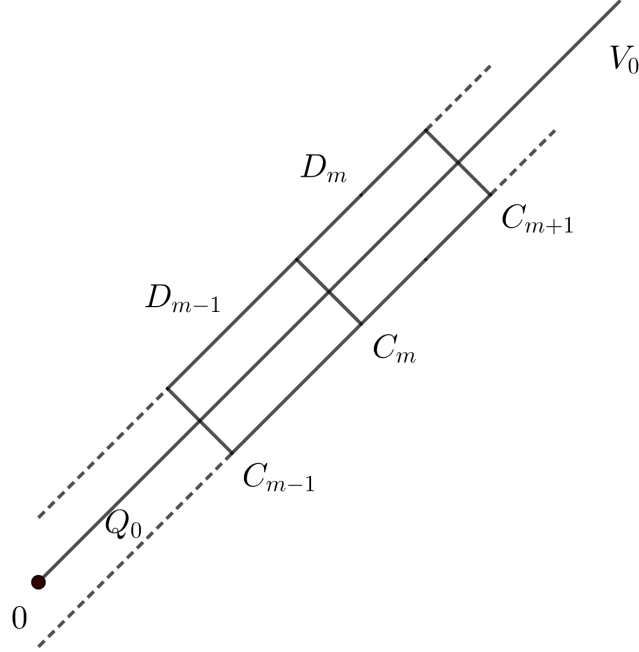
$$\arg(e^z) = \arg\left(e^{\omega_p^{p-1}z}\right), \quad (3.2.7)$$

for $z \in C_m$, $m \in \mathbb{N}$. This is important since, in the part of the plane near Q_0 , these two terms are much larger in terms of their modulus than the rest of the terms that make up the sum that defines f . For all $m \in \mathbb{N}$ we finally consider the rectangle defined by the lines C_m and C_{m+1} along with the rays $V_0 \pm (\tau / \cos(\pi/p))$, and name it D_m – see Figure 3.1.

We now describe the distribution of the zeros of f .

Theorem 3.2.5. *Let $f(z) = \sum_{k=0}^{p-1} \exp(\omega_p^k z)$ and consider the set of the rectangles $D_m \subset Q_0$ for $m \in \mathbb{N}$, as well as all their rotations around the origin that lie in Q_1, \dots, Q_{p-1} . There exists $M \in \mathbb{N}$ such that for $m > M$, there is exactly one zero of f inside each one of the p rectangles corresponding under symmetry to D_m , which additionally lies on one of the rays V_k for $k = 1, \dots, p-1$. These are the only zeros of f with a modulus larger than $M / \sin(\pi/p)$.*

Figure 3.1: The rectangles D_m and D_{m-1} .



Proof. Again, we restrict our calculations to Q_0 . We use Rouché's theorem to prove that, for large enough M , there is exactly one zero in each of the rectangles D_m for $m > M$. To that end we locate the zeros of the auxillary function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ with

$$\phi(z) = e^z + e^{\omega_p^{p-1}z}.$$

For $\phi(z) = 0$ to hold, we must have $|e^z| = |e^{\omega_p^{p-1}z}|$ and $\arg e^z = -\arg e^{\omega_p^{p-1}z}$ (or, equivalently, $u_0(z) = u_{p-1}(z)$ and $v_0(z) = -v_{p-1}(z)$). So for $z = x + iy$, and by (3.2.2) and (3.2.3), we must have

$$e^x = \exp(x \cos(2\pi/p) + y \sin(2\pi/p)),$$

which holds if and only if $y = \tan(\pi/p)x$, and we are interested in the ray with $x > 0$ (that is to say, V_0) which is contained in Q_0 . But, from (3.2.5), for $z = x + iy \in V_0$ we have

$$v_{p-1}(z) = -v_0(z) = -y,$$

and we thus get $\phi(z) = 0$ for $y = m\pi + \pi/2$ with $m \in \mathbb{N}$, since (as can easily be checked) these are exactly the points on V_0 where the arguments

of e^z and $e^{\omega_p^{p-1}z}$ cancel each other out. Hence the only zeros of ϕ in D_m are $(\cot(\pi/p)(m\pi + \pi/2), m\pi + \pi/2)$ for each $m \in \mathbb{N}$.

We now apply Rouché's theorem in D_m . We know that ϕ has exactly one zero inside each D_m . To show that the same holds for f , we shall prove the inequality

$$|f(z) - \phi(z)| < |\phi(z)| \quad (3.2.8)$$

on each ∂D_m .

Fix $m \in \mathbb{N}$. We claim that ϕ and $f - \phi$ are symmetric about V_0 as well. Then, due to symmetry, it suffices to prove inequality (3.2.8) for the part of the boundary of the rectangle that lies on and to the right-hand side of V_0 .

For ϕ , we need to show $\phi(\bar{z}e^{i\pi/p}) = \overline{\phi(ze^{i\pi/p})}$. We have

$$\phi(\bar{z}e^{i\pi/p}) = \exp(\bar{z}e^{i\pi/p}) + \exp(\bar{z}e^{-i2\pi/p}e^{i\pi/p}) = \exp(\bar{z}e^{i\pi/p}) + \exp(\bar{z}e^{-i\pi/p})$$

and

$$\overline{\phi(ze^{i\pi/p})} = \exp(\bar{z}e^{-i\pi/p}) + \overline{\exp(ze^{-i2\pi/p}e^{i\pi/p})} = \exp(\bar{z}e^{-i\pi/p}) + \exp(\bar{z}e^{i\pi/p}),$$

which proves the symmetry for ϕ .

For f , we use the function g that we previously defined in Theorem 3.2.2, with

$$g(w) = f(w^{1/p}) = p \left(1 + \frac{w}{p!} + \frac{w^2}{(2p)!} + \dots \right).$$

The function g is entire and satisfies $g(\bar{w}) = \overline{g(w)}$. But a point z and its reflection about V_0 , say z' , map to a complex conjugate pair w and \bar{w} under $z \mapsto z^p$, thus proving the symmetry about V_0 for f . Therefore both ϕ and $f - \phi$ are symmetric about V_0 as we claimed, and we can proceed with the rest of the proof.

For $z = x + iy \in \partial D_m \cap C_m$ we have, by (3.2.6) and (3.2.7),

$$|\phi(z)| = e^x + \exp(-x + 2 \cot(\pi/p)m\pi), \quad (3.2.9)$$

since, for $z = x + iy \in \partial D_m \cap C_m$,

$$\begin{aligned} \left| e^{\omega_p^{p-1}z} \right| &= \left| \exp((\cos(2\pi/p) - i \sin(2\pi/p))(x + iy)) \right| \\ &= \exp(x \cos(2\pi/p) + y \sin(2\pi/p)) \\ &= \exp \left(x \cos(2\pi/p) + \left(-\cot(\pi/p)x + \frac{m\pi}{\sin^2(\pi/p)} \right) \sin(2\pi/p) \right) \\ &= \exp(x(\cos(2\pi/p) - \cot(\pi/p) \sin(2\pi/p)) + 2m\pi \cot(\pi/p)) \\ &= \exp(-x + 2 \cot(\pi/p)m\pi). \end{aligned}$$

On this same part of ∂D_m , it is simple to see geometrically that

$$\max_{k=1,\dots,p-2} \{|\exp(\omega_p^k z)|\} = e^{u_1(z)} (= e^{u_{p-2}(z)}), \quad (3.2.10)$$

and by substituting (3.2.6) into (3.2.2) with $k = 1$ we obtain

$$\begin{aligned} u_1(z) &= x \cos\left(\frac{2\pi}{p}\right) - \left(-\cot(\pi/p)x + \frac{m\pi}{\sin^2(\pi/p)}\right) \sin\left(\frac{2\pi}{p}\right) \\ &= x (\cos(2\pi/p) + \cot(\pi/p) \sin(2\pi/p)) - 2 \cot(\pi/p) m\pi \\ &= x (\cos(2\pi/p) + 2 \cos^2(\pi/p)) - 2 \cot(\pi/p) m\pi, \end{aligned}$$

so,

$$u_1(z) = x (2 \cos(2\pi/p) + 1) - 2 \cot(\pi/p) m\pi. \quad (3.2.11)$$

Now, for $z = x + iy$ in this same part of ∂D_m , we can write $x = m\pi \cot(\pi/p) + c$, with $c \in (0, \tau \sin(\pi/p))$ (as this is the perpendicular, and $\tau \sin(\pi/p)$ denotes the horizontal distance from a point in V_0 to ∂D_m). Hence, to verify (3.2.8) for $z \in D_m \cap C_m$, by (3.2.9), (3.2.10) and (3.2.11), it suffices to show that

$$\exp(m\pi \cot(\pi/p) + c) + \exp(m\pi \cot(\pi/p) - c) \quad (3.2.12)$$

is greater than $(p-2)e^{u_1(z)}$ for all $c \in (0, \tau \sin(\pi/p))$, which we can write as

$$(p-2) \exp((m\pi \cot(\pi/p) + c) (2 \cos(2\pi/p) + 1) - 2 \cot(\pi/p) m\pi),$$

by (3.2.11), that is,

$$(p-2) \exp(m\pi \cot(\pi/p) (2 \cos(2\pi/p) - 1) + c(2 \cos(2\pi/p) + 1)). \quad (3.2.13)$$

By Lemma 3.2.4, for

$$\begin{aligned} d &= (p-2) \exp(c(2 \cos(2\pi/p) + 1)) / (e^c + e^{-c}), \\ x &= m\pi \cot(\pi/p), \text{ and} \\ a &= 2 \cos(\pi/p) - 1, \end{aligned}$$

we deduce that (3.2.12) is greater than (3.2.13) for large enough $m \in \mathbb{N}$, and we let M be the largest integer for which it is not.

For the side of ∂D_m which lies in C_{m+1} we can repeat the above calculations for $m+1$ instead of m .

It remains to show the inequality (3.2.8) for the side of ∂D_m which lies in $\partial T_0(\nu)$.

From (2.0.3) we can deduce that

$$|e^z| \geq \sum_{k=1}^{p-1} \left| e^{\omega_p^k z} \right|$$

for $z \in \partial T_0(\nu)$. Hence

$$\begin{aligned} |\phi(z)| &= \left| e^z + e^{\omega_p^{p-1} z} \right| \\ &\geq |e^z| - \left| e^{\omega_p^{p-1} z} \right| \\ &\geq \sum_{k=1}^{p-1} \left| e^{\omega_p^k z} \right| - \left| e^{\omega_p^{p-1} z} \right| \\ &= \sum_{k=1}^{p-2} \left| e^{\omega_p^k z} \right| \\ &\geq \left| \sum_{k=1}^{p-2} e^{\omega_p^k z} \right| \\ &= |f(z) - \phi(z)|. \end{aligned}$$

Thus, by Rouché's theorem, we have proven that for $m > M$, the number of zeros of f in the corresponding rectangle D_m is one, and we know this zero must lie on V_0 by Theorem 3.2.2. We now show that for sufficiently large values of m , the unique zero of f in D_m lies between $y = m\pi$ and $y = (m+1)\pi$.

For the two dominant terms e^z and $e^{\omega_p^{p-1} z}$, and for points $z = x + iy \in V_0$, from (3.2.4) and (3.2.5) we have

$$u_0(z) = u_{p-1}(z) = x$$

and

$$v_0(z) = -v_{p-1}(z) = y.$$

Hence, for points $z = x + iy \in V_0$, $m \in \mathbb{N}$, we get, by Lemma 3.2.3, the first terms of the sum of f to be $2e^x$ for $y = 2m\pi$, and $-2e^x$ for $y = 2m\pi + \pi$. From Lemma 3.2.4 we deduce that for all $p \geq 3$, there exists $x_0 > 0$ such

that for $x > x_0$ and $z = x + iy \in V_0$ (so $y = x \tan(\pi/p)$),

$$\begin{aligned} e^x &> (p-2) \max_{k=1,2,\dots,p-2} \exp(u_k(z)) \\ &= (p-2) \max_{k=1,2,\dots,p-2} \exp\left(\frac{x}{\cos(\pi/p)} \cos\left(\frac{\pi(2k+1)}{p}\right)\right) \\ &= (p-2) \exp\left(\frac{x}{\cos(\pi/p)} \cos\left(\frac{3\pi}{p}\right)\right), \end{aligned}$$

since this inequality is equivalent to $E_{p-2,a}(x) > 0$ for $a = \cos(3\pi/p)/\cos(\pi/p)$. Thus,

$$f(z) > 0 \text{ for } z = x + iy \in V_0 \text{ with } y = m\pi, \ m \in \mathbb{N},$$

and

$$f(z) < 0 \text{ for } z = x + iy \in V_0 \text{ with } y = (m+1)\pi, \ m \in \mathbb{N}.$$

It follows from the intermediate value theorem that, for all sufficiently large $m \in \mathbb{N}$, f vanishes at some point in each of the segments between the points $y = m\pi$ and $y = m\pi + \pi$ on V_0 , and thus this is the only zero of f inside the corresponding rectangle D_m .

The more general result, as stated, follows from obvious symmetry arguments and from the fact that the distance of D_M from the origin is $M/\sin(\pi/p)$. \square

In the following, we will use Theorem 3.2.5 to locate the critical points of f .

Theorem 3.2.6. *Let $f \in \mathcal{F}$. All critical points of f lie in $\cup_{k=0,\dots,p-1} V_k$ and are separated in each V_k from each other by the zeros of f . Further, all the critical values of f lie on the real axis, alternating between the positive and negative axes.*

Proof. Let h be the principal branch of $z \mapsto z^{1/p}$. Recall that, as in Theorem 3.2.2,

$$g(w) = f(w^{1/p}) = p \left(1 + \frac{w}{p!} + \frac{w^2}{(2p)!} + \dots \right),$$

so $g = f \circ h$ and the function g is entire. We investigate its order:

$$\begin{aligned} \rho(f \circ h) &= \limsup_{r \rightarrow \infty} \frac{\log(\log M(r, f \circ g))}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log(\log M(r^{1/p}, f))}{\log r} \end{aligned}$$

and, for $s = r^{1/p}$,

$$\begin{aligned}\rho(f \circ h) &= \limsup_{s \rightarrow \infty} \frac{\log(\log M(s, f))}{p \log s} \\ &= \frac{1}{p} \rho(f) = \frac{1}{p},\end{aligned}$$

since

$$M(r, f \circ h) = \max\{|f(z^{1/p})| : |z| = r\} = \max\{|f(z)| : |z| = r^{1/p}\} = M(r^{1/p}, f).$$

We state a theorem by Laguerre [61, p. 266].

Theorem 3.2.7. *If f is an entire function, real for real z , of order less than 2, with only real zeros, then the zeros of f' are also all real, and are separated from each other by the zeros of f .*

Since g is real on \mathbb{R} , we can apply Laguerre's theorem to $g = f \circ h$, which certainly has order less than 2, and whose zeros all lie on the negative real axis, from the proof of Theorem 3.2.2. We deduce that the critical points of $g = f \circ h$ are all real and are separated by the zeros of g . Hence, by symmetry, the critical points of f lie on $\cup_{k=0}^{p-1} V_k$ and on each V_k they are separated by the zeros of f .

Further, since the rays V_k are mapped under f onto the real axis, the critical values of f all lie on the real axis. \square

3.3 Trapeziums

In this section we locate compact subsets of the plane that cover themselves under f . Inside these compact sets we construct invariant Cantor sets for f on which f is conjugate to the one-sided shift on Σ_N , and which will serve as the endpoints of the curves in the Cantor bouquet. Specifically, these compact sets are the trapeziums $T_{m,c}$ bounded by the lines

$$\begin{aligned}y &= \tan(\pi/p)x \\ y &= (2m-1)\pi \\ y &= (2m+1)\pi \\ x &= c,\end{aligned}$$

for $m \in \mathbb{N}$, for large enough $c > \tau/\sin(\pi/p)$, and with their sides labelled, respectively, as $S_{m,c}^1, S_{m,c}^2, S_{m,c}^3, S_{m,c}^4$. Note that we define the sets $T_{m,c}$ to contain the boundary as well as the inside of each trapezium. We also define $T_{-m,c}$ to be the reflection of $T_{m,c}$ with respect to \mathbb{R} .

We know from Theorem 3.2.5 that, for large enough m , f maps $S_{m,c}^1$ into a bounded subset of the real axis which contains 0 (which f attains twice on $S_{m,c}^1$).

We proceed to investigate the curves $f(S_{m,c}^2)$ and $f(S_{m,c}^3)$. We give a lemma about the behaviour of f at the points on the half-lines $Y_m^\pm := \{x + iy \in \mathbb{C} : y = (2m \pm 1)\pi, x > \cot(\pi/p)y\}$ for $m \in \mathbb{N}$, which are the horizontal half-lines that contain a segment of the boundary of the trapezium $T_{m,c}$ (analogous results immediately follow for $T_{-m,c}$).

Lemma 3.3.1. *There exists $M \in \mathbb{N}$ such that $f(Y_m^\pm)$ lies in the left half-plane for all $m > M$.*

Proof. Let $z = x + iy$. We prove the result for Y_m^+ ; Y_m^- is similar. For $x + iy \in Y_m^+$, we will write $x = \cot(\pi/p)y_m + \xi$ where $y_m = (2m + 1)\pi$ and $\xi > 0$. Assuming that $x + iy \in Y_m$ in the following, we have, from (3.2.2) and (3.2.3), for $k = 0, 1, \dots, p-1$,

$$u_k(z) = \frac{y_m}{\sin(\pi/p)} \cos\left(\frac{\pi(2k+1)}{p}\right) + \xi \cos\left(\frac{2k\pi}{p}\right) \quad (3.3.1)$$

and

$$v_k(z) = \frac{y_m}{\sin(\pi/p)} \sin\left(\frac{\pi(2k+1)}{p}\right) + \xi \sin\left(\frac{2k\pi}{p}\right), \quad (3.3.2)$$

which we now treat as functions of ξ . For $k \in \{0, \dots, p-1\}$ we define the functions $\phi_k : (0, +\infty) \rightarrow \mathbb{R}$ by

$$\phi_k(\xi) = \exp(u_k(z)) \cos(v_k(z)),$$

so,

$$\operatorname{Re} f(z) = \sum_{k=0}^{p-1} \exp(u_k(z)) \cos(v_k(z)) = \sum_{k=0}^{p-1} \phi_k(\xi).$$

We want to prove that $\operatorname{Re} f(z) < 0$ for all points in Y_m when m is sufficiently large. The two largest terms with respect to their moduli for small ξ are the terms corresponding to $k = 0$ and to $k = p-1$. By (3.3.1) and (3.3.2) these are, respectively,

$$\phi_0(\xi) = -\exp(y_m \cot(\pi/p) + \xi), \quad (3.3.3)$$

which is negative for all $\xi > 0$, and (since $\sin(\pi(2(p-1)+1)/p) = -\sin(\pi/p)$ and $\sin(2(p-1)\pi/p) = -\sin(2\pi/p)$)

$$\phi_{p-1}(\xi) = \exp(y_m \cot(\pi/p) + \xi \cos(2\pi/p)) \cos(y_m + \xi \sin(2\pi/p)),$$

which is negative if ξ satisfies

$$\pi < \pi + \xi \sin(2\pi/p) < 3\pi/2;$$

that is, if

$$0 < \xi < \frac{\pi}{2 \sin(2\pi/p)}.$$

But for $\xi \geq \pi/(2 \sin(2\pi/p))$ we have

$$\begin{aligned} \left| \frac{\phi_{p-1}(\xi)}{\phi_0(\xi)} \right| &\leq \frac{\exp(y_m \cot(\pi/p) + \xi \cos(2\pi/p))}{\exp(y_m \cot(\pi/p) + \xi)} \\ &= \exp(\xi(\cos(2\pi/p) - 1)), \end{aligned}$$

which is a function of ξ that decreases to 0 as ξ increases to ∞ , and thus attains its maximum value for $\xi = \pi/(2 \sin(2\pi/p))$. There consequently exists $\mu = \mu(p) \in (0, 1)$ such that

$$|\phi_{p-1}(\xi)| \leq \mu |\phi_0(\xi)|, \text{ for } \xi \geq \pi/(2 \sin(2\pi/p)),$$

and thus, since $\phi_0(\xi)$ is negative for all $\xi > 0$,

$$\phi_0(\xi) + \phi_{p-1}(\xi) \leq (1 - \mu)\phi_0(\xi), \text{ for all } \xi > 0. \quad (3.3.4)$$

It is simple to see that, for all $\xi > 0$,

$$\max_{k=1, \dots, p-2} |\phi_k(\xi)| = |\phi_1(\xi)| = \exp\left(\frac{y_m \cos(3\pi/p)}{\sin(\pi/p)} + \xi \cos(2\pi/p)\right), \quad (3.3.5)$$

so

$$\sum_{k=1}^{p-2} |\phi_k(\xi)| \leq (p-2) |\phi_1(\xi)|, \text{ for } \xi > 0.$$

We now use Lemma 3.2.4. Following its notation, we substitute

$$x = y_m \cot(\pi/p),$$

as well as

$$d = 2(p-2) \exp(\xi(\cos(2\pi/p) - 1))/(1 - \mu)$$

and

$$a = \cos(3\pi/p) / \cos(\pi/p) < 1,$$

and choose M so large that $e^x - de^{ax} > 0$ for all $m > M$. Returning to the original notation of this part, it follows that for all $m > M$ and $x + iy \in Y_m$, we have

$$(p-2)|\phi_1(\xi)| \leq \frac{1-\mu}{2}|\phi_0(\xi)|,$$

by (3.3.3) and (3.3.5), and thus

$$\sum_{k=1}^{p-2} |\phi_k(\xi)| \leq \frac{1-\mu}{2} |\phi_0(\xi)| \text{ for } \xi > 0, \quad (3.3.6)$$

by (3.3.5) again. Hence, finally, for $z \in Y_m$ with $m > M$, from (3.3.4) and (3.3.6) we have

$$\operatorname{Re} f(z) = \sum_{k=0}^{p-1} \phi_k(\xi) \leq \frac{1-\mu}{2} \phi_0(\xi) < 0 \text{ for } \xi > 0,$$

as required. \square

From Lemma 3.3.1, we know that $f(S_{m,c}^2)$ and $f(S_{m,c}^3)$ lie in the left half-plane for $m > M$. We now investigate the behaviour of $f(S_{m,c}^4)$ in order to complete the proof that the trapeziums cover themselves under f . Specifically, we prove the following.

Lemma 3.3.2. *For large enough $c > 0$, the set $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\} \cap f(S_{m,c}^4)$ is a curve that meets both the positive and negative imaginary axes and lies inside an annulus of the form*

$$A_{m,c} := \{z : ||z| - e^c| \leq \max_{z \in S_{m,c}^4} |(f - \exp)(z)|\}.$$

Proof. The image of $S_{m,c}^4$ under e^z is the circle around the origin with radius e^c . Additionally, by Lemma 2.0.7 and by (3.2.2) (since $S_{m,c}^4 \subset T_0(\nu)$),

$$\begin{aligned} \max_{z \in S_{m,c}^4} |(f - \exp)(z)| &= \max_{z \in S_{m,c}^4} \sum_{k=1}^{p-1} \exp(u_k(z)) \\ &\leq (p-1) \max_{z \in S_{m,c}^4} \exp(u_{p-1}(z)) \\ &= (p-1) \max_{z \in S_{m,c}^4} \exp \left(c \cos \left(\frac{2\pi}{p} \right) + y \sin \left(\frac{2\pi}{p} \right) \right) \\ &= (p-1) \exp \left((2m+1)\pi \sin \left(\frac{2\pi}{p} \right) \right) \exp \left(c \cos \left(\frac{2\pi}{p} \right) \right). \end{aligned}$$

But, since $\cos(2\pi/p) < 1$, the quantity $\max_{z \in S_{m,c}^4} |(f - \exp)(z)|$ is small compared to e^c , since

$$\frac{\exp(c)}{\exp(c \cos(2\pi/p))} = \exp(c(1 - \cos(2\pi/p))) \rightarrow \infty, \text{ as } c \rightarrow \infty.$$

Thus, for large enough c , $f(z)$ for $z \in S_{m,c}^4$ has to lie inside a ball of radius $\max_{z \in S_{m,c}^4} |(f - \exp)(z)|$ around a point on $\{z : |z| = e^c\}$. For large enough c , then, $f(S_{m,c}^4)$ is a curve inside an annulus that meets both the positive and negative imaginary axes, and joins up the rest of the image of $\partial T_{m,c}$ under f in the left half-plane. \square

We can now define inverse branches of f inside the trapeziums $T_{m,c}$, using the following lemma.

Lemma 3.3.3. *There exists $c > 0$ such that the inverse branch of f in $T_{m,c}$ is well defined and analytic for large enough m .*

Proof. Let $r(m, c)$ denote the radius of the inner boundary curve of the annulus $A(m, c)$ that was defined in Lemma 3.3.2; that is,

$$r(m, c) := e^c - \max_{z \in S_{m,c}^4} |(f - \exp)(z)|.$$

Fix $K \in \mathbb{N}$. Choose $c_0 > 1$ such that $T_{i,c}$ and $T_{j,c}$ exist for $c \geq c_0$, with $T_{i,c} \subset f(T_{j,c})$ for all $i, j \in \{M, \dots, M + K - 1, -M, \dots, -M - K + 1\}$; Lemmas 3.3.1 and 3.3.2 guarantee that such a $c_0 > 1$ exists. The image curve $f(S_{m,c}^4)$ goes round $A(m, c)$, so it surrounds $\{z : \operatorname{Re} z > 0\} \cap B(0, r(m, c))$. We can thus deduce from Rouché's theorem that the points inside $\{z : \operatorname{Re} z > 0\} \cap B(0, r(c, m))$ are covered by the image of $T_{m,c}$ under f as many times as they would be covered under \exp ; that is, once. Hence, the inverse of f in $T_{m,c}$ is well defined. \square

The values M and K will remain fixed from now on, and for the following we assume that $c = c_0$ and write $T_{M+j,c} = T_{j+1}$, $T_{-M-j,c} = T_{-j-1}$ for $j \in \{0, \dots, K - 1\}$ and $S_{m,c}^i = S_m^i$. Let L_j be the branch of the inverse of f on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cap B(0, r(c, m))$ that takes values in T_j .

Let $T^K = \cup_{1 \leq |j| \leq K} T_j$ and let Λ_K be the set of points whose orbits remain for all time in T^K (see Figure 1). Recall that the trapeziums are closed sets.

Theorem 3.3.4. *The set Λ_K is homeomorphic to Σ_K and $f|_{\Lambda_K}$ is conjugate to the shift map on Σ_K .*

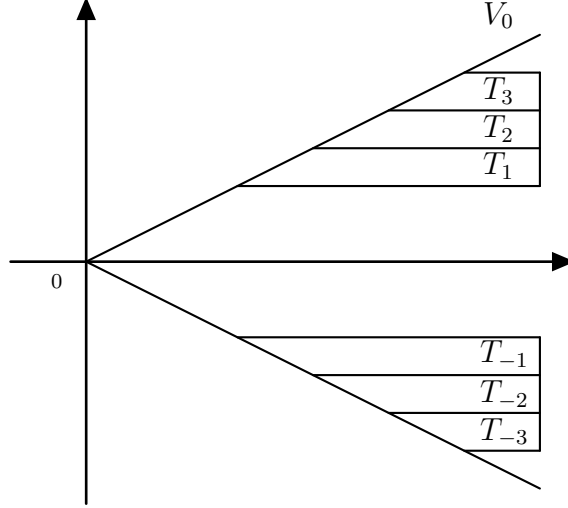


Figure 3.2: The set T^K for $K = 3$.

Proof. Let $s = s_0 s_1 s_2 \dots \in \Sigma_K$ and define

$$L_s^n(z) = L_{s_0} \circ \dots \circ L_{s_{n-1}}(z), \quad \text{for } z \in T^K. \quad (3.3.7)$$

We claim that, for $z \in T^K$,

$$\lim_{n \rightarrow \infty} L_s^n(z)$$

exists and is independent of z .

For any j with $1 \leq |j| \leq K$, consider the image set $f(\text{int } T_j)$; from Lemmas 3.3.1 and 3.3.2 and the remarks before them it follows that this image set will lie inside $B(0, r(c, m))$ and cover T^K . This allows us to choose a simply connected region G inside $f(\text{int } T_j)$ for any j with $1 \leq |j| \leq K$, so that $T^K \subset G$. Thus, for each j , there exists an open connected subset of T_j , say T'_j , which maps univalently onto G under f . Then the inverse branch L_{s_j} maps T'_j strictly inside itself and so each L_{s_j} is a strict contraction with respect to the Poincaré metric on G . In particular, each L_{s_j} is a uniformly strict contraction with respect to the Poincaré metric on $T^{K'} = \cup_{1 \leq |j| \leq K} T'_j$

(since there are only finitely many L_{s_j}). Therefore, the sets $L_s^n(T^{K'})$ are nested and their diameters decrease to 0 as $n \rightarrow \infty$. So $\lim_{n \rightarrow \infty} L_s^n(z)$ exists and is independent of z .

We can thus define $\Phi(s) = \lim_{n \rightarrow \infty} L_s^n(z)$ for all $z \in T^{K'}$. A standard argument (see, for example, [17, Theorem 9.9]) then shows that Φ is a homeomorphism, which gives the conjugacy between f and the shift map. \square

For each $s \in \Sigma_K$, let $z(s)$ be the unique point in Λ_K whose itinerary is s .

Corollary 3.3.5. *Let $s = \overline{s_0 s_1 \cdots s_{n-1}}$ be a repeating sequence in Σ_K . Then $z(s)$ is a repelling periodic point of f with period n .*

Proof. The map L_s^n is a composition of analytic maps and therefore analytic itself. Also, $L_s^n(T_{s_0})$ is contained in the interior of T_{s_0} . Since L_s^n is a strict contraction with respect to the Poincaré metric on T_{s_0} , it follows that T_{s_0} has a unique fixed point in this trapezium and that this fixed point is attracting for L_s^n ; thus repelling for f . Since this point has itinerary s for f , it must be $z(s)$. \square

Corollary 3.3.6. *Let $s \in \Sigma_K$. Then $z(s) \in J(f)$.*

Proof. From Corollary 3.3.5 it follows that $z(s)$ is a limit of repelling periodic points given by the conjugacy with the shift map. By a result of Baker [3], $J(f)$ is the closure of the set of repelling periodic points of f ; hence $z(s) \in J(f)$. \square

We note that $z(s)$ is not the only point inside the strip $R(s_0)$ that has itinerary s : in fact, there are infinitely many points in this strip that share the itinerary s and form a curve, as we will show in the next section.

To end this section, we prove an additional property of the points $z(s)$, $s \in \Sigma_K$; some of them will always exist in $R(m) \cap Q_0$ and $R(-m) \cap Q_1$ for large enough positive m .

Lemma 3.3.7. *Let $f \in \mathcal{F}$. For $m \geq M$, there exist $s, s' \in \Sigma_K$, such that $z(s) \in R(m) \cap Q_0$ and $z(s') \in R(-m) \cap Q_1$.*

Proof. Consider the repeating sequence $s = \overline{s_j}$ in Σ_K . Then $z(s) \in R(j)$. If $z(s)$ was in $T_0(\nu)$, from Lemma 2.0.1 we would have $f^n(z(s)) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Therefore, $z(s) \in Q_0 \cup Q_1$. \square

3.4 Hairs

We continue to use the proof strategy of [18] in this section, with our goal being to show that each point in Λ_K (constructed in the previous section, see Theorem 3.3.4) actually lies at the endpoint of a curve, all points of which share the same itinerary.

Definition 3.4.1. Let $s = s_0 s_1 s_2 \dots \in \Sigma_K$. A curve $h_s : [1, \infty) \rightarrow R_{s_0}$ is called a *hair* of f that is attached to $z(s)$ if

1. $h_s(1) = z(s)$;
2. for each $t \geq 1$, the itinerary of $h_s(t)$ under f is s ;
3. if $t > 1$, then $\lim_{n \rightarrow \infty} \operatorname{Re} f^n(h_s(t)) = \infty$;
4. $\lim_{t \rightarrow \infty} \operatorname{Re} h_s(t) = \infty$;

or, if it is a rotation of a curve that satisfies the above properties.

In the following we continue to focus our attention on $T_\nu(0)$, since analogous results follow for the other sectors due to symmetry. A hair attached to $z(s)$ is a curve that extends from the endpoint $z(s)$ to infinity in the right half-plane. Any point z on this hair that is not $z(s)$, shares the same itinerary as $z(s)$, and has an orbit that tends to infinity in the right half-plane. Further, the orbit of the endpoint $z(s)$ is bounded since $s \in \Sigma_K$.

To show that such hairs exist, we need a covering result which we can obtain as a corollary to two lemmas from the previous section.

Corollary 3.4.2. *Let $H_{m,c}$ be the half-strip that extends to infinity in the right half-plane, bounded by the lines*

$$\begin{aligned} y &= (2m - 1)\pi \\ y &= (2m + 1)\pi \\ x &= c, \end{aligned}$$

for $m \in \mathbb{N}$ and $c > 0$. Then, for large enough $c > 0$, we have

$$\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cap f(H_{m,c}) \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cap B(0, R(c))^c,$$

where $R(c) \sim e^c$ as $c \rightarrow \infty$.

Proof. This is an immediate consequence of Lemmas 3.3.1 and 3.3.2, keeping in mind that we can apply Lemma 3.3.2 to all large enough c . \square

Following the notation established in the previous section, we extend the inverse branches of f , L_j (previously defined in the proof of Theorem 3.3.4), to the half-strips $H_{m,c}$ as follows:

Let $z \in H_{m,c}$ for some m and c . Then $L_j(z)$ is the preimage under f of z in $H_{j,c}$. Note that, for all m and c and any j , we have

$$\left| L'_j(z) \right| < 1/2 \text{ for } z \in H_{m,c}, \quad (3.4.1)$$

by (2.0.4).

We will now prove that if $s = s_0 s_1 s_2 \dots$ is a bounded sequence, then there is a unique hair attached to $z(s)$. Let $E(z) = (1/e)e^z$. For any $s \in \Sigma_K$ we define the functions $G_s^n : [1, \infty) \rightarrow \mathbb{C}$ by

$$G_s^n(t) = L_s^n \circ E^n(t), \quad 1 \leq t < \infty;$$

recall that L_s^n was defined in (3.3.7). We will show that the limit function of G_s^n exists and that it provides a parametrisation of the hair h_s as a function of t . First, an inequality about the functions $G_s^n(t)$.

Proposition 3.4.3. *There exist $q, M > 0$ such that, for any $s = s_0 s_1 s_2 \dots \in \Sigma_K$ and for all $t > q + 1$ and $n \geq 1$, we have*

$$t - M \leq \operatorname{Re} G_s^n(t) \leq t + M. \quad (3.4.2)$$

Proof. Since $|s_i| \leq K$ for all i , there exists $M_K > 2\pi$ such that $|\operatorname{Im} L_{s_i}(z)| < M_K$ for each s_i and all $z = x + iy \in T_0(\nu)$, whose preimages we will consider.

Define $\epsilon(z) := f(z)/e^z - 1$. We will make use of the following estimate of the quantity $|1 + \epsilon(z)|$:

$$\begin{aligned} |1 + \epsilon(z)| &= \left| \frac{e^z + \sum_{k=1}^{p-1} \exp(\omega_p^k z)}{e^z} \right| \\ &= \left| 1 + \sum_{k=1}^{p-1} \exp((\omega_p^k - 1)z) \right| \\ &\leq 1 + (p-1) \max_{k \in \{1, \dots, p-1\}} |\exp((\omega_p^k - 1)z)| \\ &\leq 1 + (p-1) \max_{k \in \{1, \dots, p-1\}} \exp(\operatorname{Re}((\omega_p^k - 1)z)) \\ &\leq 1 + (p-1) \exp \left(\max_{k \in \{1, \dots, p-1\}} ((\cos(2k\pi/p) - 1)x - y(\sin(2k\pi/p) - 1)) \right). \end{aligned}$$

We can thus write

$$|1 + \epsilon(z)| \leq 1 + C e^{ax}, \quad (3.4.3)$$

where

$$a = \max_{k \in \{1, \dots, p-1\}} (\cos(2k\pi/p) - 1) < 0$$

and

$$C = (p-1) \max_{k \in \{1, \dots, p-1\}, |y| \leq M_K} \exp(-y(\sin(2\pi k/p) - 1)) > 0.$$

Recall that $E : \mathbb{R} \rightarrow \mathbb{R}$ is $E(t) = (1/e)e^t$. We define

$$s_n(t) := \sum_{k=0}^n \log(1 + C \exp(aE^k(t)))$$

for $t > 1$, with $s(t) := \lim_{n \rightarrow \infty} s_n(t)$. The series defining $s(t)$ is convergent and the function s is decreasing to 0 with respect to t , since $a < 0$ and $C = C(p, k)$ is independent of y .

In the following we will consider several lower bounds for q , starting here: we can choose $q > 0$ large enough and, further, $M = M(q) > 1$ such that

$$s_n(t) \leq s(t) \leq M - 1 \tag{3.4.4}$$

and

$$s_n(t) \leq s(t) + \log M_K - 1 \leq M, \tag{3.4.5}$$

for all $t > q + 1$. We further choose $q > 0$ to be large enough so that

$$E(q) > M + 2 \tag{3.4.6}$$

and consider all $t > q + 1$ such that

$$\operatorname{Re} L_j(E(t)) \geq q \tag{3.4.7}$$

for all $|j| \leq K$.

For $t \geq 1$, we have $E(t) \geq 1$, so

$$1 + \frac{s_n(q)}{E(t)} \leq e^{s_n(q)}, \quad \text{for all } n \in \mathbb{N} \text{ and } q > 0. \tag{3.4.8}$$

For the rest of the proof we assume that q is large enough so that (3.4.4), (3.4.5), (3.4.7) and (3.4.8) all hold. We will prove that for any sequence $s \in \Sigma_{K'}$ (with $K' \leq K - 1$) it is the case that

$$t - M \leq \operatorname{Re} G_s^m(t)$$

for all $n \in \mathbb{N}$ and all such large enough $t > q + 1$.

We have

$$f(z) = e^z(1 + \epsilon(z)),$$

so

$$e^z = \frac{f(z)}{1 + \epsilon(z)}$$

and thus, if $f(z) = w$, with $z, w \in T_0(\nu)$, then the corresponding inverse branch is

$$f^{-1}(w) = z = \log w - \log(1 + \epsilon(z)) \quad (3.4.9)$$

for the appropriate branch of the logarithm. Hence we can write

$$\begin{aligned} \operatorname{Re} G_s^1(t) &= \operatorname{Re} L_{s_0}(E(t)) \\ &= \operatorname{Re} \log E(t) - \operatorname{Re} \log(1 + \epsilon(L_{s_0}(E(t)))) \\ &\geq t - 1 - \log(1 + Ce^{aq}) \\ &\geq t - 1 - s_0(q) \\ &\geq t - M \end{aligned}$$

with the first inequality following from (3.4.3) and (3.4.7), the second following from the fact that $s_0(q) \leq s(q)$, while the third follows from (3.4.4). This is the first step of the induction. Now let us assume that, for all $s \in \Sigma_k$, for some $m \geq 3$ and for all q large enough and $t > E(q)$, we have

$$\operatorname{Re} G_s^m(t) \geq t - 1 - s_{m-1}(q), \quad (3.4.10)$$

from which

$$\operatorname{Re} G_s^m(t) \geq t - M$$

follows, by (3.4.4) and the definition of $s(q)$. We will proceed to deduce that (3.4.10) holds with m replaced by $m + 1$. Substituting $E(t)$ for t and $E(q)$ for q in (3.4.10), we obtain

$$\operatorname{Re} G_s^m(E(t)) \geq E(t) - 1 - s_{m-1}(E(q))$$

from which it follows by (3.4.4) and the definition of $s(q)$ that

$$\operatorname{Re} G_s^m(E(t)) \geq E(t) - M. \quad (3.4.11)$$

So, by (3.4.3) and (3.4.9),

$$\begin{aligned}
\operatorname{Re} G_s^{m+1}(t) &= \operatorname{Re} L_{s_0} (G_{\sigma(s)}^m(E(t))) \\
&\geq \log |G_{\sigma(s)}^m(E(t))| - \log(1 + Ce^{aq}) \\
&\geq \log |\operatorname{Re} G_{\sigma(s)}^m(E(t))| - \log(1 + Ce^{aq}) \\
&= \log (\operatorname{Re} G_{\sigma(s)}^m(E(t))) - \log(1 + Ce^{aq}),
\end{aligned}$$

with the last equality following from (3.4.6) and (3.4.11). We claim that

$$\begin{aligned}
\operatorname{Re} G_s^{m+1}(t) &\geq \log (E(t) - 1 - s_{m-1}(E(q))) - \log(1 + Ce^{aq}) \\
&\geq \log E(t) - \log(1 + s_{m-1}(E(q))) - \log(1 + Ce^{aq}) \\
&\geq \log E(t) - s_{m-1}(E(q)) - \log(1 + Ce^{aq}) \\
&= \log E(t) - s_m(q) \\
&\geq t - 1 - s_m(q) \\
&\geq t - M,
\end{aligned}$$

for any $n \in \mathbb{N}$ and $t > E(q)$. The third inequality follows easily from the fact that $\log(1 + x) \leq x$ for $x > 0$. We now prove that the second one holds as well.

Here we use the fact that, for $a > b > 1$ and $a \geq b^2/(b - 1)$, we have

$$\log(a - b) \geq \log a - \log b,$$

which we apply with $a = E(t)$ and $b = 1 + s_{m-1}(E(q))$. For $t > E(q)$, we have

$$E(t) > 1 + s_{m-1}(E(q)) > 1,$$

so it remains to show that, for large enough q , (and since $t > E(q)$)

$$E(E(q)) \geq \frac{(1 + s_{m-1}(E(q)))^2}{s_{m-1}(E(q))},$$

or, equivalently,

$$E(E(q)) \geq 1/s_{m-1}(E(q)) + 2 + s_{m-1}(E(q)).$$

The quantity $s_{m-1}(E(q))$ decreases to 0 as q increases to infinity, but

$$1/s_{m-1}(E(q)) \leq 1/s_0(E(q)) = 1/\log(1 + Ce^{aE(q)}).$$

For q large enough we have $Ce^{aE(q)} \leq 1$, since C is bounded and $-1 < a < 0$, so

$$\frac{1}{\log(1 + Ce^{aE(q)})} \leq \frac{Ce^{-aE(q)}}{\log 2} \leq E(E(q)),$$

again for q large enough, and using the fact that

$$\frac{\log(1+x)}{x} \geq \log 2$$

for $0 < x < 1$.

We have thus proven our claim that (3.4.10) holds with $m+1$ replacing m and so the induction is complete.

We will now, again using induction, prove that

$$\operatorname{Re} G_s^m(t) \leq t + M$$

for all $n \geq 1$, and for q large enough and $t > q + 1$. From (3.4.3) and (3.4.4) we have

$$\begin{aligned} \operatorname{Re} G_s^1(t) &= \operatorname{Re} L_{s_0}(E(t)) \\ &= \operatorname{Re} \log E(t) - \operatorname{Re} \log(1 + \epsilon(L_{s_0}(E(t)))) \\ &\leq t - 1 + \log(1 + Ce^{aq}) \\ &\leq t + M. \end{aligned}$$

Now suppose that

$$\operatorname{Re} G_s^m(t) \leq t + s_m(q),$$

so, in particular,

$$\operatorname{Re} G_u^s(t) \leq t + M.$$

Then,

$$\operatorname{Re} G_{\sigma(s)}^m(E(t)) \leq E(t) + \sum_{k=1}^m \log(1 + C \exp(aE^k(q))),$$

and

$$\operatorname{Re} G_s^{m+1}(t) = \operatorname{Re} L_{s_0}(G_{\sigma(s)}^m(E(t))).$$

From this, together with (3.4.9), we have

$$\operatorname{Re} G_s^{m+1}(t) \leq \log |G_{\sigma(s)}^m(E(t))| + \log(1 + Ce^{aq}).$$

Thus,

$$\begin{aligned} \operatorname{Re} G_s^{m+1}(t) &\leq \log(|\operatorname{Re} G_{\sigma(s)}^m(E(t))| + |\operatorname{Im} G_{\sigma(s)}^m(E(t))|) + \log(1 + Ce^{aq}) \\ &\leq \log(\operatorname{Re} G_{\sigma(s)}^m(E(t))) + \log(1 + Ce^{aq}) + \log M_K, \end{aligned}$$

since $\log(a+b) \leq \log a + \log b$ as long as $a, b > 2$ (recall that $\operatorname{Re} G_{\sigma(s)}^m(E(t)) \geq E(t) - M > 2$ from (3.4.6) and (3.4.11), as well as that $M_K > 2\pi$). Now, by taking logarithms in (3.4.8), we have

$$\begin{aligned} \operatorname{Re} G_s^{m+1}(t) &\leq \log \left(E(t) + \sum_{k=1}^m \log(1 + C \exp(aE^k(q))) \right) + \log(1 + Ce^{aq}) + \log M_K \\ &\leq t - 1 + \sum_{k=1}^m \log(1 + C \exp(aE^k(q))) + \log(1 + Ce^{aq}) + \log M_K \\ &= t - 1 + \sum_{k=0}^m \log(1 + C \exp(aE^k(q))) + \log M_K \\ &= t - 1 + s_m(q) + \log M_K \\ &\leq t + M, \end{aligned}$$

thus proving the desired result for $m+1$ and completing the induction. \square

We now prove that

$$h_s(t) := \lim_{n \rightarrow \infty} G_s^n(t)$$

is a well defined function for $t \geq 1$. It suffices to prove that $\{G_s^n(t)\}$ is Cauchy for all large t . From the result of Proposition 3.4.3 and the quantity M_K defined at the start of its proof, we have, for large enough t ,

$$|G_s^n(t) - G_s^{n+1}u(t)| \leq 2(M + M_K)$$

for any $s \in \Sigma_K$. For those large enough t we have

$$\begin{aligned} |G_s^{N+n}(t) - G_s^{N+n+1}(t)| &= |L_s^N \circ G_{\sigma^N(s)}^n(t) - L_s^N \circ G_{\sigma^N(s)}^{n+1}(t)| \\ &\leq |(L_s^N)'(z)| |G_{\sigma^N(s)}^n(t) - G_{\sigma^N(s)}^{n+1}(t)| \\ &\leq (1/2)^N 2(M + M_K), \end{aligned}$$

with the last inequality due to (3.4.1). Now let $\varepsilon > 0$. There exists $N = N(\varepsilon) > 0$ such that, for all $m > n \geq N$,

$$|G_s^{N+n}(t) - G_s^{N+m}(t)| \leq 2(M + M_K) \sum_{k=0}^{m-n-1} \frac{1}{2^{N+k}},$$

which is less than ε for large enough N . This proves our claim that h_s is well defined.

Next, we prove that h_s is continuous in $[1, \infty)$. We initially leave out $t = 1$; it is handled separately below.

Proposition 3.4.4. *Suppose that $s = s_0 s_1 s_2 \dots \in \Sigma_K$. Then $h_s(t)$ is continuous as a function of $t \in (1, \infty)$.*

Proof. Choose α with $0 < \alpha < 1$ and let q and M be as specified in the previous proposition. Choose $T > q + 2M$ so that, if $\operatorname{Re} z > T$ and $|\operatorname{Im} z| < M_K$ (with M_K defined as in Proposition 3.4.3), then

$$|L'_{s_i}(z)| < \alpha. \quad (3.4.12)$$

This is possible due to (2.0.4) of Lemma 2.0.1. By Proposition 3.4.3, if $t > T$, then

$$E^k(t) - M \leq \operatorname{Re} G_s^n(E^k(t)) \leq E^k(t) + M, \quad (3.4.13)$$

for all $n, k \geq 0$.

We first prove the continuity of $h_s(t)$ for $t > T$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ so that $\alpha^k(3M + 2\pi) < \varepsilon$. Given $t_0 > T$, choose δ such that, if $|t - t_0| < \delta$, then $|E^k(t) - E^k(t_0)| < M$. We claim that, if $|t - t_0| < \delta$, then $|h_s(t) - h_s(t_0)| < \varepsilon$. Indeed, we note that for such t and each $n \geq 0$ we have

$$|G_{\sigma^k(s)}^n(E^k(t)) - G_{\sigma^k(s)}^n(E^k(t_0))| < 3M + 2\pi.$$

This follows since, by (3.4.13) and our choice of δ ,

$$\left| \operatorname{Re} G_{\sigma^k(s)}^n(E^k(t)) - \operatorname{Re} G_{\sigma^k(s)}^n(E^k(t_0)) \right| < |E^k(t) - E^k(t_0)| + 2M < 3M$$

and

$$\left| \operatorname{Im} G_{\sigma^k(s)}^n(E^k(t)) - \operatorname{Im} G_{\sigma^k(s)}^n(E^k(t_0)) \right| < 2\pi.$$

Consequently, by (3.4.12) and (3.4.13) for $|t - t_0| < \delta$ and $n \geq 0$,

$$\begin{aligned} |G_s^{n+k}(t) - G_s^{n+k}(t_0)| &= \left| L_s^k \circ G_{\sigma^k(s)}^n(E^k(t)) - L_s^k \circ G_{\sigma^k(s)}^n(E^k(t_0)) \right| \\ &\leq \alpha^k \left| G_{\sigma^k(s)}^n(E^k(t)) - G_{\sigma^k(s)}^n(E^k(t_0)) \right| \\ &\leq \alpha^k(3M + 2\pi) < \varepsilon, \end{aligned}$$

from which it follows that $t \mapsto h_s(t)$ is continuous for any $s = s_0 s_1 s_2 \dots \in \Sigma_K$ and $t > T$.

We will now prove continuity for $1 < t \leq T$. If $1 < t < T$, then there exists k (depending on t) such that $E^k(t) > T$. Then, by the earlier part of the proof,

$$t \mapsto L_s^k \circ h_{\sigma^k(s)}(E^k(t))$$

is continuous, since each inverse function of f is well defined and continuous on the half-strips $H_{m,c}$; see the remark following the proof of Corollary 3.4.2. But this map is given by

$$t \mapsto \lim_{n \rightarrow \infty} L_s^k \circ G_{\sigma^k(s)}^n \circ E^k(t) = h_s(t),$$

and since k depends only on t , the result follows. \square

We now prove continuity for $t = 1$.

Proposition 3.4.5. *Suppose that $s \in \Sigma_K$. Then $h_s(t)$ is continuous at $t = 1$.*

Proof. From Section 3.3, we know that, for all $i \geq 0$, $L_{s_i}(z)$ maps T^K (defined in that section as the union of the relevant trapeziums) strictly inside itself for any c large enough, with $x = c$ being the line on which the right-hand sides of each of the trapeziums in T^K lie. As we previously saw in Lemma 3.3.4, L_{s_i} is a strict contraction with respect to the Poincaré metric on T^K . We need to use this fact to prove our result, but also we want to benefit from the inequalities of Proposition 3.4.3.

To that end, note that for a value of t that is sufficiently close to 1, there exists some integer N (dependent on t) such that $E^N(t)$ is larger than the q that is specified in Proposition 3.4.3. Thus, for any $n > N$, we have $G_{\sigma^n(s)}^n(E^N(t)) \in T^K$ for c sufficiently large. We can now use the Poincaré metric to show that the distance between $h_s(1) = z(s)$ and

$$h_s(t) = \lim_{N \rightarrow \infty} L_s^N \circ G_{\sigma^n(s)}^n(E^N(t))$$

can be made arbitrarily small as $t \rightarrow 1^+$.

To prove this, first we note that the endpoint $z(s)$ lies in $T_{s_0,c}$ for any sufficiently large $c > 0$. Choose $c > 0$ to be large enough so that

1. $f(T_{j,c})$ contains T_c^K for all $1 \leq |j| \leq K$, and
2. all endpoints for the hairs corresponding to itineraries $s = s_0 s_1 s_2 \dots$ with $|s_j| \leq K$ lie in T_c^K .

Then, $z(s)$ lies in $T_{j,c}$. Since $z(\sigma^k(s))$ lies in $T_{s_k,c}$, we claim that $z(s)$ arises as the limit of successive preimages under f of the trapeziums (in accordance with the itinerary).

To prove this claim, we use the Branner–Hubbard criterion (see, for example, [41, p. 233, Problem 2.5]). Let us denote by $\text{mod } A$ the conformal modulus of an open topological annulus $A \subset \mathbb{C}$. The Branner–Hubbard criterion states that if $K_1 \supset K_2 \supset K_3 \supset \dots$ is a nested sequence of compact connected subsets of \mathbb{C} , with each K_{n+1} contained in the interior of K_n , and if, further, each interior K_n° is simply connected (making each difference $A_n = K_n^\circ \setminus K_{n+1}$ a topological annulus), then if $\sum_{n=1}^\infty \text{mod } A_n = \infty$, the intersection $\cap K_n$ reduces to a single point. The sets A_n in our case, are the difference of the starting set with the corresponding preimage under f ; that is, $A_1 = T_{s_0,c} \setminus \overline{L_{s_0}(T_{s_1,c})}$ and $A_n = L_{s_0} \circ \dots \circ L_{s_{n-2}}(T_{s_{n-1},c}) \setminus \overline{L_{s_0} \circ \dots \circ L_{s_{n-1}}(T_{s_n,c})}$ for $n \geq 2$. But f maps conformally between these trapeziums (since f is entire on \mathbb{C} and the zeros of f' lie on the lines $\cup_{k=0,\dots,n-1} V_k$ as shown in Theorem 3.2.6), making the conformal moduli in each step constant and thus proving our claim. \square

Finally, we prove that to $z(s)$, for each $s \in \Sigma_K$, there corresponds a *unique* curve that is attached to it and is parametrised by $t \mapsto h_s(t)$.

Theorem 3.4.6. *Let $s = s_0 s_1 s_2 \dots \in \Sigma_K$. There is a unique hair attached to $z(s)$ and $t \mapsto h_s(t)$ is a parametrisation of this hair. In particular, this hair lies entirely in $R(s_0)$.*

Proof. We first verify that h_s is indeed a hair, following Definition 3.4.1. We claim that $h_s(t)$ has itinerary s for $t \geq 1$. Note that, since $f \circ L_{s_0}$ is the identity, we have, for $t \geq 1$,

$$f \circ h_s(t) = \lim_{n \rightarrow \infty} f \circ G_s^n(t) = \lim_{n \rightarrow \infty} G_{\sigma(s)}^{n-1}(E(t)) = h_{\sigma(s)}(E(t)).$$

It follows that, for $t \geq 1$,

$$f^n \circ h_s(t) = h_{\sigma^n(s)}(E^n(t)). \quad (3.4.14)$$

Hence $f^n(h_s(t)) \in R(s_n)$ (which denotes the horizontal 2π -width strip that corresponds to s_n) as required. Also, from Proposition 3.4.3 and (3.4.14),

$$E^n(t) - M \leq \text{Re } f^n \circ h_s(t) \leq E^n(t) + M,$$

for n sufficiently large, where M is as specified in Proposition 3.4.3. Therefore, $\text{Re } f^n \circ h_s(t) \rightarrow \infty$ as $n \rightarrow \infty$ when $t > 1$. Finally, since $t - M \leq \text{Re } h_s(t) \leq t + M$ for $t > q$, it follows that $\text{Re } h_s(t) \rightarrow \infty$ as $t \rightarrow \infty$. This proves that h_s parametrises a hair. We will now show that this hair is unique.

Suppose that h_s is not unique. Then there are at least two hairs attached to $z(s)$; consider two of them. We examine the following two cases.

- Suppose that the hairs meet in only a bounded set of points. Consider the last point of intersection; suppose that point is ζ . Let U be the unbounded open set consisting of the set of points contained in $R(s_0)$ that is bounded by the two hairs, has ζ in its boundary and can only access infinity from the right half-plane. We claim that the images of U under f^n are contained within the images of the hairs attached to $f^n(\zeta)$ (so, in $T_0(\nu)$) and therefore, by Montel's theorem, U has to be in the Fatou set of f , which contradicts Lemma 2.0.4.

To prove the claim, we consider a point $z \in U$ and intersect U with the half-plane $\{w \in \mathbb{C} : \operatorname{Re} w < \lambda\}$, with $\lambda > \operatorname{Re} z$; name the new bounded set U_λ . The crosscuts of U_λ on the vertical $\{w \in \mathbb{C} : \operatorname{Re} w = \lambda\}$ map under \exp to arcs of a circle around the origin of radius e^λ . When λ is large enough, f will map the crosscuts to a thin annulus around that circle. The set $f(U_\lambda)$ can then be one of two bounded sets defined by $f(\partial U_\lambda)$ inside some circle around the origin. But, since the two hairs have to lie in $R(s_1)$ following the itinerary s and $f(U_\lambda)$ does not contain the origin, $f(U_\lambda)$ is a bounded region that is defined between them and has to lie in $R(s_1)$ as well. Its further forward images will lie in their respective strips in $T_0(\nu)$, thus avoiding the left half-plane. Since $\lambda > \operatorname{Re} z$ was arbitrary, the forward images under f of the unbounded region U also have to also lie in $T_0(\nu)$, thus proving our claim.

- Suppose that the curves meet in an unbounded set of points. Since the hairs are unbounded closed sets that are not identical, there must exist a domain U lying in $R(s_0)$ whose boundary lies entirely in the two hairs. As in the previous case, the images of U under f^n are bounded by the images of the hairs attached to $f^n(\zeta)$ and therefore, by Montel's theorem, U is in the Fatou set of f . This contradicts Lemma 2.0.4. \square

We can now deduce our main result.

Proof of Theorem 3.1.1. Let $f \in \mathcal{F}$. Theorem 3.4.6 gives the existence of a Cantor bouquet in $T_0(\nu) \cup Q_0 \cup Q_1$. Consider an arbitrary hair of the Cantor bouquet, parametrised by $t \mapsto h_s(t)$. From Lemma 3.3.6, $z(s)$ is in $J(f)$. Let $z = h_s(t)$ with $t > 1$. Then $f^n(z) \rightarrow \infty$ in the sector $T_0(\nu)$, so there exists $N \in \mathbb{N}$ such that for $n \geq N$, $f^n(z) \in T_0(\nu)$. Therefore, from Lemma 2.0.4, $z \in J(f) \cap A(f)$. \square

Finally, we have the following corollary of Theorem 3.4.6 and Lemma 3.3.7, which reveals a key part of the structure of the Julia set given by Theorem 3.4.6.

Corollary 3.4.7. *Let $f \in \mathcal{F}$. For all large enough $k \in \mathbb{N}$, there exist two simple unbounded curves γ_k and γ_{-k} in $J(f)$, with their endpoints in Q_0 and Q_1 respectively, that lie entirely inside the strips $R(k)$ and $R(-k)$ respectively, and tend to infinity through $T_0(\nu)$.*

We note that the symmetry properties of the function f allow us to extend the result of this corollary to $T_k(\nu)$, for $k = 1, \dots, p-1$.

Chapter 4

Absence of wandering domains

4.1 Statements of main results

The study conducted in Chapter 3 illuminates aspects of the structure of the Julia and escaping sets of functions in the family \mathcal{F}_p , $p \geq 3$, defined in (2.0.1). In particular, we proved that there exist Cantor bouquets in each of these functions' Julia set spiders' webs (see Theorem 3.1.1).

In this chapter we prove that for a subfamily of these functions, there exist no wandering domains. Further, for many of them, the Julia set is the whole plane. We state our results.

Theorem 4.1.1. *Let $f_\lambda \in \mathcal{F}_p$, $p \geq 3$, with p even and $\lambda \in \mathbb{R}^*$. Then f has no wandering domains.*

Using Theorem 4.1.1, we can also prove the following.

Theorem 4.1.2. *Let $f_\lambda \in \mathcal{F}_p$, $p \geq 3$, with p even and $|\lambda| \geq 1$. Then $J(f) = \mathbb{C}$.*

In order to prove our results, we use a new technique based on a result on the relationship between wandering domains and points in the postsingular set, proved in 2017 by Barański, Fagella, Jarque and Karpińska [9, Theorem B], which was stated as Theorem 1.3.7 in the introduction. We restate it here for the reader's convenience.

Theorem 4.1.3. *Let f be a transcendental meromorphic map and U be a Fatou component of f . Denote by U_n the Fatou component such that $f^n(U) \subset U_n$. Then, for every $z \in U$, there exists a sequence (p_n) in $P(f)$ such that*

$$\frac{\text{dist}(p_n, U_n)}{\text{dist}(f^n(z), \partial U_n)} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

In particular, if for some $d > 0$ we have $\text{dist}(f^n(z), \partial U_n) < d$ for all n , then $\text{dist}(p_n, U_n) \rightarrow 0$ as n tends to ∞ .

We note that the methods discussed here cannot be applied in the case where p is odd, as the situation is more delicate in that case, with the postsingular set lying in a part of the plane where we do not have good control of the dynamics.

The structure of this chapter is as follows:

- In Section 4.2, we discuss some preliminary results and extra properties of the functions in \mathcal{F}_p .
- Section 4.3 contains the proofs of Theorems 4.1.1 and 4.1.2, along with an example that demonstrates that there exist small values of λ for which the result of Theorem 4.1.2 does not hold.

4.2 Preliminaries

Recall that Sixsmith studied in [59] the class \mathcal{E}_p of transcendental entire functions defined for $p \geq 3$ as

$$\mathcal{E}_p = \left\{ f : f(z) = \sum_{k=0}^{p-1} a_k \exp(\omega_p^k z), \text{ where } a_p \in \mathbb{C}^* \text{ for } k \in \{0, 1, \dots, p-1\} \right\},$$

where $\omega_p = \exp(2\pi i/p)$ is an p th root of unity. In this chapter we will restrict our studies to the family \mathcal{E}_p , $p \geq 3$, where $a_i \in \mathbb{R}$ and $a_i = a_j$ for all $i, j \in \{0, 1, \dots, p-1\}$; that is, the family \mathcal{F}_p as defined in Section 2.1. An advantage of this restriction is that we have strong control over points in $P(f)$ for \mathcal{F}_p (as will be seen in results quoted in this section from Chapter 3), which we lack for the broader class \mathcal{E}_p . This control is essential in order to apply Theorem 4.1.3.

Recall the partition of the plane discussed in Chapter 2. A key result, Lemma 2.0.4, concerns the behaviour of f in $T_j(\nu)$, $j \in \{0, 1, \dots, p-1\}$. For our purposes here, we quote only a small part of that lemma:

Lemma 4.2.1. *Let $f \in \mathcal{E}_p$, $p \geq 3$. There exist $\nu' > 0$ and $\varepsilon_0 \in (0, 1)$ such that, for all $z \in T(\nu)$ with $\nu \geq \nu'$,*

$$|f(z)| > \max\{e^{\varepsilon_0 \nu}, M(\varepsilon_0 |z|, f)\}. \quad (4.2.1)$$

We need the location of the zeros and critical points of $f \in \mathcal{F}_p$, $p \geq 3$. In Chapter 3, we defined rays that lie inside the strips Q_k :

Definition 4.2.2. Let $V_0 := \{x + iy \in \mathbb{C} : y = \tan(\pi/p)x, x > 0\}$ and let its $2k\pi/p$ -rotations clockwise around the origin for $k = 1, \dots, p-1$ be V_1, \dots, V_{p-1} .

The following result follows from Theorems 3.2.2 and 3.2.6:

Lemma 4.2.3. *Let $f \in \mathcal{F}_p$, $p \geq 3$. The following hold.*

- *All zeros of f lie in $\cup_{k=0, \dots, p-1} V_k$.*
- *$CP(f) \subset \cup_{k=0, \dots, p-1} V_k$, and critical points are separated from each other in each V_k by the zeros of f .*
- *$CV(f) \subset \mathbb{R}$.*

The next result on the location of the postsingular set for functions in \mathcal{F}_p follows easily from the results above.

Corollary 4.2.4. *Let $f \in \mathcal{F}_p$, $p \geq 3$. Then $\overline{P(f)} \subset \mathbb{R}$.*

Proof. From Theorem 2.0.3, $A_R(f)$ is a spider's web, so f has no asymptotic values [52, Theorem 1.8]. From Lemma 4.2.3, all the critical values of f lie in \mathbb{R} . But f is real on \mathbb{R} , so $f^n(CV(f)) \subset \mathbb{R}$ for all $n \in \mathbb{N}$. Therefore $\overline{P(f)} \subset \mathbb{R}$. \square

We now prove an elementary result for the functions $f \in \mathcal{F}_p$, $p \geq 3$, for even p .

Lemma 4.2.5. *Let $f \in \mathcal{F}_p$, $p \geq 3$, where p is even, and let ν' be as in Lemma 4.2.1. There exists $r_0 > 0$ such that if*

$$S_r = \{x + iy : |x| \geq r, |y| \leq \pi/2p\},$$

then, for all $r \geq r_0$, $|f(z)| > |z|$ for all $z \in S_r$ and $f(S_r) \subset T_0(\nu')$.

Proof. Let $z \in S_r$. From [59, Lemma 4.1], there exists $\epsilon(r)$ with $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$, such that $f(z) \in B(e^z, \epsilon(r))$; observe that this implies that $|f(z)| > |z|$ for all $z \in S_r$ for large enough r .

Recall from Chapter 2.1 that $T_0(\nu')$ is part of the sector $\{te^{i\phi} : t > 0, |\phi| \leq \pi/p\}$. But, for sufficiently large r , if $z \in S_r$, then $|\arg e^z| \leq \pi/2p$, so we have $f(z) \in B(e^z, \epsilon(r)) \subset T_0(\nu')$. Thus $f(S_r) \subset T_0(\nu')$ for all large r . \square

Recall from Chapter 2.2 that we defined the following strips of width 2π for all $k \in \mathbb{Z}$:

$$R(k) := \{z \in \mathbb{C} : (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi\}.$$

Finally, we prove a lemma which allows us to use Theorem 4.1.3 for any function $f \in \mathcal{F}_p$, $p \geq 3$, with ease.

Lemma 4.2.6. *Let $f \in \mathcal{F}_p$, $p \geq 3$. Then there exists $c > 0$ such that $\text{dist}(z, J(f)) \leq c$ for all $z \in F(f)$.*

Proof. Fix $\nu \geq \nu'$, where ν' is as given in Lemma 4.2.1. Due to symmetry, it suffices to prove the result for points in the Fatou set that lie in $P(\nu) \cup T_0(\nu) \cup Q_0 \cup Q_1$ (with $P(\nu)$ as defined in Chapter 2.1; see also Figure 2.1).

Suppose $z \in P(\nu)$. Since $P(\nu)$ is a bounded region around the origin, the result holds as the Julia set is non-empty.

On the other hand, if $z \in Q_0 \cup Q_1 \cup T_0(\nu)$, then the result follows immediately from Lemma 3.3.7. \square

4.3 Proof of Theorem 4.1.1

Let $f \in \mathcal{F}_p$ for some $p \geq 3$ with p even. We will show that f has no wandering domains. Fix $\nu = \nu'$, where ν' is as given in Lemma 4.2.1. Suppose that U is a wandering domain for f , and put $U_n = f^n(U)$ for $n \geq 0$. Since U is bounded (as $J(f)$ is a spider's web from Theorem 2.0.3), it follows that each U_n is a Fatou component.

Let $z \in U$. It follows from Lemma 4.2.6 that there exists $c > 0$ such that $\text{dist}(f^n(z), \partial U_n) \leq c$ for all $n \in \mathbb{N}$. It then follows from Theorem 4.1.3 that there exists a sequence (p_n) in $P(f)$, such that

$$\text{dist}(p_n, U_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

From Corollary 4.2.4, we have $P(f) \subset \mathbb{R}$, so

$$\text{dist}(U_n, \mathbb{R}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.3.1)$$

We will show that the above properties imply that, for all large enough $n \in \mathbb{N}$, U_n has to lie in $T_0(\nu)$; thus giving a contradiction to Lemma 2.0.4.

To that end: from Definition 1.6.1, since $J(f)$ is a spider's web, we can find a domain G such that the following hold (as illustrated in Figure 4.1):

- $\partial G \subset J(f)$
- $\partial G \cap \gamma_k \neq \emptyset$ and $\partial G \cap \gamma_{-k} \neq \emptyset$,
- $G \supset \{z : |x| = r_0, |y| \leq \pi/2p\}$, and
- $P(\nu) \subset G$,

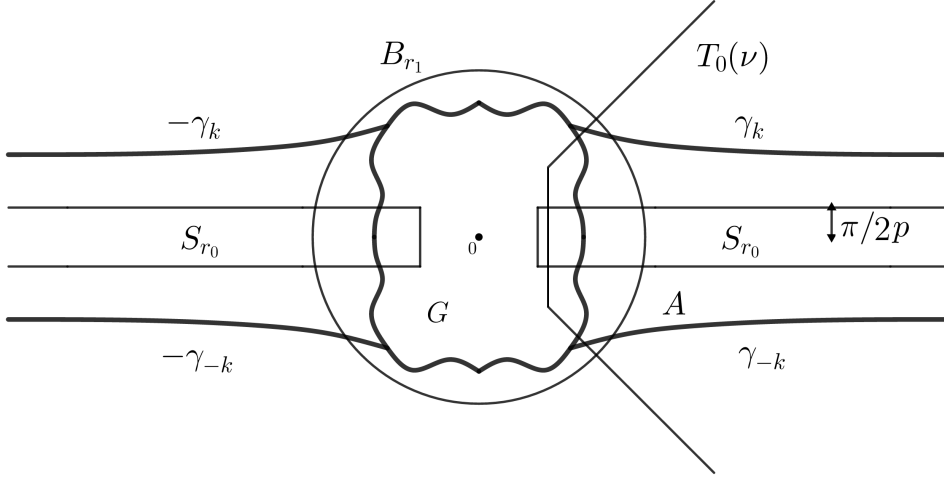


Figure 4.1: Some of the sets described in the proof of Theorem 4.1.1. The bold curves are in the Julia set.

where γ_k and γ_{-k} are the curves in $J(f)$ defined in Corollary 3.4.7 for some $k \in \mathbb{N}$, and r_0 is large enough so that the result of Lemma 4.2.5 holds. We also define the curves $-\gamma_k$ and $-\gamma_{-k}$ to be the reflections about the y -axis of γ_k and γ_{-k} respectively. They also lie in $J(f)$ as f is even.

We define A to be the unbounded region in $T_0(\nu)$ with $\partial A \subset \partial G \cup \gamma_k \cup \gamma_{-k} \subset J(f)$ and $A \cap \mathbb{R} \neq \emptyset$ (see Figure 4.1).

Let B_{r_1} denote the disk around the origin of radius r_1 , where r_1 is large enough so that $G \subset B_{r_1}$ and

$$|f(z)| > |z| \text{ for } z \in T_0(\nu) \cap B_{r_1}^c; \quad (4.3.2)$$

this is possible by (4.2.1).

It follows from Theorem 2.0.3 and Corollary 1.6.7 that points in U do not have bounded orbit. Also, it follows from (4.3.1) that there exists $n_0 \in \mathbb{N}$ such that

$$\text{dist}(U_n, \mathbb{R}) \leq \pi/p \text{ for all } n \geq n_0. \quad (4.3.3)$$

Let n_0 also be such that U_{n_0} lies outside B_{r_1} ; this is possible because U_n has to lie between two consecutive loops of the spider's web for each $n \in \mathbb{N}$ and U does not have bounded orbit. Thus, by Lemma 4.2.5, we can take $w_0 \in S_{r_0} \cap U_{n_0} \cap B_{r_1}^c$ such that $f(w_0) \in T_0(\nu')$ and $|f(w_0)| > |w_0|$; in particular, $f(w_0)$ is also outside B_{r_1} . So

$$f(w_0) \in T_0(\nu) \cap B_{r_1}^c \cap U_{n_0+1}. \quad (4.3.4)$$

We know from (4.3.3) that $U_{n_0+1} \cap \{z : |\text{Im } z| \leq \pi/p\} \neq \emptyset$, and, since

$\partial G \cup \pm\gamma_k \cup \pm\gamma_{-k} \subset J(f)$ and $f(w_0) \in U_{n_0+1}$, it follows from (4.3.4) that

$$U_{n_0+1} \cap S_{r_0} \cap \{z : \operatorname{Re} z > 0\} \neq \emptyset.$$

We thus have

$$(i) \quad U_{n_0+1} \cap B_{r_1}^c \neq \emptyset$$

$$(ii) \quad U_{n_0+1} \cap S_{r_0} \cap \{z : \operatorname{Re} z > 0\} \neq \emptyset.$$

These two properties together imply that $U_{n_0+1} \subset A \subset T_0(\nu)$. We now show that if properties (i) and (ii) are satisfied for U_m (for some $m \geq n_0 + 1$), they will also be satisfied for U_{m+1} .

Figures 4.2 to 4.6 help to illustrate this proof. The square, the triangle and the star in these figures denote the points z_1 , z_2 and z_3 respectively, along with part of their orbits under f .

Suppose then that (i) and (ii) are satisfied for U_m , for some $m \geq n_0 + 1$; we have $z_1 \in U_m \cap B_{r_1}^c$ and $z_2 \in U_m \cap S_{r_0} \cap \{z : \operatorname{Re} z > 0\}$ (and hence $U_m \subset A \subset T_0(\nu)$) – see Figure 4.2 and Figure 4.3.

Property (i) is immediately satisfied for U_{m+1} , as $f(z_1) \in U_{m+1} \cap B_{r_1}^c$ by (4.3.2). Since $z_2 \in S_{r_0}$, we have $f(z_2) \in U_{m+1} \cap T_0(\nu)$ by Lemma 4.2.5. See Figure 4.4.

Since $f(z_1) \in B_{r_1}^c$, we have $U_{m+1} \subset \mathbb{C} \setminus G$. By (4.3.3), there exists $z_3 \in U_{m+1} \cap \{z : |\operatorname{Im} z| \leq \pi/p\}$. Since $U_{m+1} \subset \mathbb{C} \setminus G$, we have $z_3 \in S_{r_0}$. Since $f(z_2) \in U_{m+1} \cap T_0(\nu)$ by Lemma 4.2.5, we have $z_3 \in S_{r_0} \cap \{z : \operatorname{Re} z > 0\}$ (using the same reasoning as above), thus satisfying property (ii) – see Figures 4.5 and 4.6.

Since properties (i) and (ii) together imply that $U_m \subset A \subset T_0(\nu)$, it follows by induction that $U_m \subset A \subset T_0(\nu)$ for all $m \geq n_0 + 1$, giving a contradiction to Lemma 2.0.4. Thus our supposition that U is a wandering domain was false.

Figure 4.2: We start with $z_1 \in U_m \cap B_{r_1}^c$ and $z_2 \in U_m \cap S_{r_0} \cap \{z : \operatorname{Re} z > 0\}$.

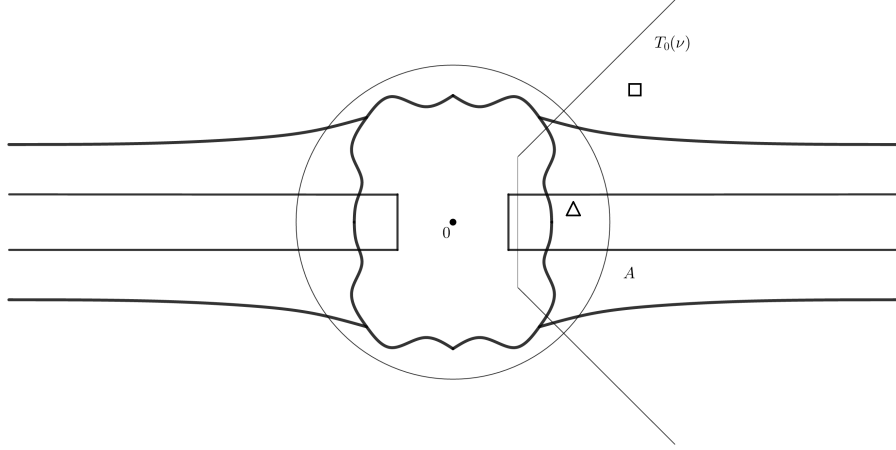


Figure 4.3: Both z_1 and z_2 have to be in A , since the bold lines are in the Julia set.

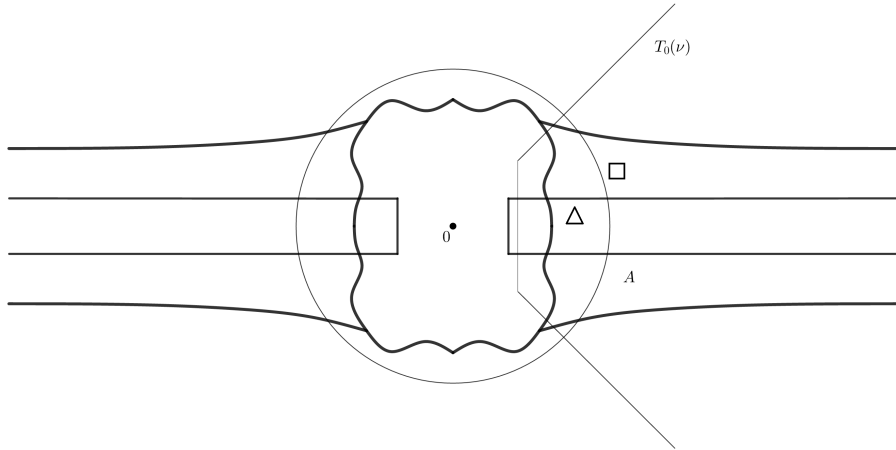


Figure 4.4: We have $f(z_1) \in U_{m+1} \cap B_{r_1}^c$ and $f(z_2) \in U_{m+1} \cap T_0(\nu)$.

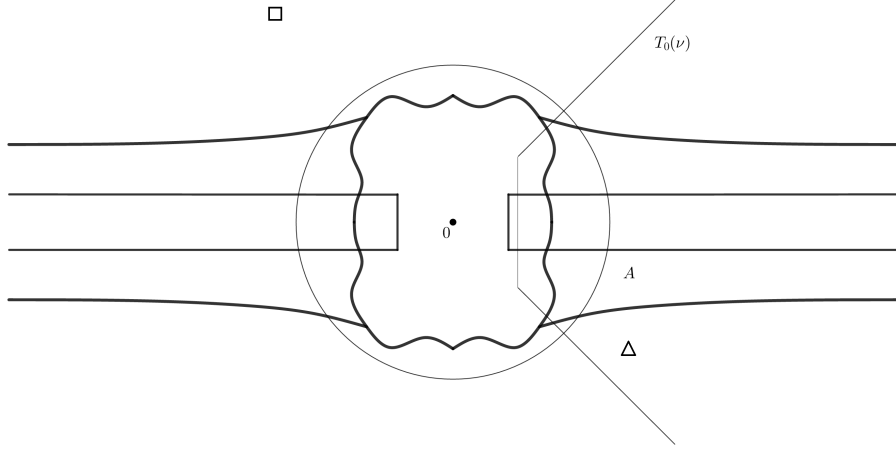


Figure 4.5: Since the bold lines are in $J(f)$, and z_3 and $f(z_2)$ are in U_{m+1} , we have $z_3 \in S_{r_0} \cap \{z : \operatorname{Re} z > 0\}$.

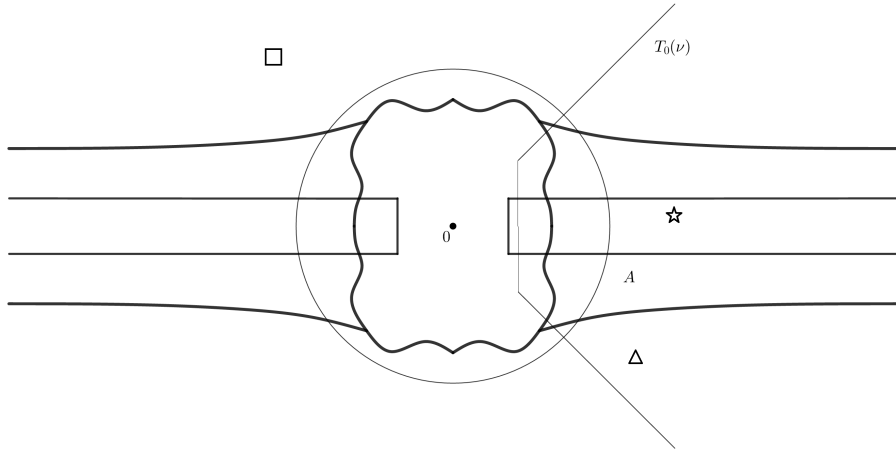
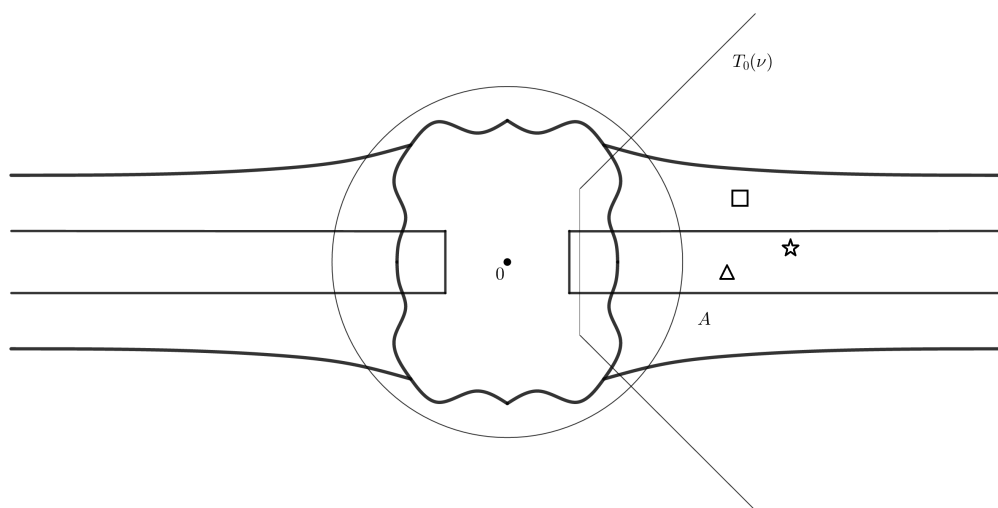


Figure 4.6: Since z_3 is in A and the bold lines are in the Julia set, the whole component has to lie in A . Therefore $f(z_1)$ and $f(z_2)$ also have to be in A . The point $f(z_1)$ satisfies property (i) of the induction, while z_3 satisfies property (ii).



4.4 Proof of Theorem 4.1.2

We now prove that, if p is even and $|\lambda| \geq 1$, the Julia set of $f = f_\lambda \in \mathcal{F}_p$ is the whole plane. We offer the proof for $\lambda \geq 1$. The proof for $\lambda \leq -1$ is similar since f is even as a function when p is even.

Proof. Let U be a Fatou component of f . From Theorem 4.1.1, U cannot be a wandering domain. Without loss of generality, for the rest of the proof we assume that U is periodic (otherwise we could work with $f^j(U)$ for some $j > 0$).

From Theorem 2.0.3 we have that $J(f)$ is a spider's web, so U is bounded and, in particular, cannot be a Baker domain. The remaining cases are that U belongs to an attracting cycle, a parabolic cycle, or is a Siegel disk. In each of these cases, we have $\overline{U} \cap P(f) \neq \emptyset$ [10, Theorem 7]. From Corollary 4.2.4, we have $P(f) \subset \mathbb{R}$.

For large $x > 0$ we have $f(x) > 0$, and since from Lemma 4.2.3 there are no zeros of f in \mathbb{R} (as they lie on the rays V_0, \dots, V_{p-1} and $f(0) = \lambda p$), we have $f(x) > 0$ for all $x \in \mathbb{R}$. Further, we claim that $f(x) > x$ for all $x \in \mathbb{R}$. Since p is even, we just need to prove this for $x \geq 0$.

Suppose that $\lambda = 1$, which makes the function values the smallest possible within our range of values of λ . From Theorem 3.2.2, for $x \geq 0$ we have

$$f(x) = p \left(1 + \frac{x^p}{p!} + \frac{x^{2p}}{(2p)!} + \dots \right),$$

and thus

$$\frac{f(x)}{x} \geq \frac{p}{x} \left(1 + \frac{x^p}{p!} \right) = g(x).$$

Now, we have

$$g'(x) = -\frac{p}{x^2} + \frac{x^{p-2}}{(p-2)!} = 0$$

if and only if

$$x^p = p(p-2)!.$$

Hence, g has a unique minimum on $(0, \infty)$ with value

$$\frac{p}{(p(p-2)!)^{1/p}} \left(1 + \frac{p(p-2)!}{p!} \right) = \frac{p}{(p(p-2)!)^{1/p}} \left(1 + \frac{1}{p-1} \right) > 1,$$

since $p^p > p(p-2)!$.

Therefore, $\mathbb{R} \subset I(f)$. Since $\overline{P(f)} \subset \mathbb{R}$, we have $\overline{P(f)} \subset I(f)$. Thus, $\overline{U} \cap I(f) \neq \emptyset$, which is a contradiction since U is a bounded periodic Fatou component. \square

Even though $|\lambda| \geq 1$ is not the sharpest value for the result of Theorem 4.1.2 to hold, in the following proposition we demonstrate that there do exist small values of λ for which the result does not hold.

Proposition 4.4.1. *Let $f = f_{1/4} \in \mathcal{F}_4$. There exists an attracting Fatou basin for f .*

Proof. For $\lambda = 1/4$ and $p = 4$ we can write

$$f(z) = \frac{1}{2}(\cos z + \cosh z).$$

We have $f(0) = 1$, while

$$f(\pi/2) = \frac{1}{2}(0 + \cosh(\pi/2)) \approx 1.25,$$

so $f(\pi/2) - \pi/2 < 0$.

Since, for $x \geq 0$, we have

$$f(x) = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \dots,$$

the function f is real, increasing and convex on $[0, \infty)$. We thus deduce that there exists an attracting fixed point of f in $(0, \pi/2)$, which has to lie in an attracting Fatou basin. \square

Chapter 5

Future work

In this chapter we discuss questions that have arisen from our research and are of interest. The questions range from immediate and specific to more general, as in the last subsection. We also provide some partial results where possible.

5.1 Wandering domains for the odd subfamily

In Chapter 4 (Theorem 4.1.1) we showed that, if $f \in \mathcal{F}_p$ for some $p \geq 3$ with p even, then f has no wandering domains. We stated that our methods cannot be used in the case that p is odd. We now discuss this in more detail.

Recall that the critical values of f lie on the real axis (Lemma 4.2.3). If p is even, most of the real axis is contained in the regions $T_0(\nu)$ and $T_{p/2}$ where we have good control of the dynamics (following Lemma 2.0.1).

If p is odd, however, then the negative real axis is contained in the strip $Q_{(p+1)/2}$, where we do not have good control. It is therefore theoretically possible for the wandering domain to reside near the negative real axis.

Question 5.1.1. Can functions in \mathcal{F}_p have wandering domains when p is odd?

We have a partial result in this direction which states that, if wandering domains do exist for functions in \mathcal{F}_p , where p is odd, they have to eventually lie in a strip around the negative real axis. To prove it, we use a distortion theorem for escaping points [10, Lemma 7, p. 165].

Lemma 5.1.2. *Let g be analytic in \mathbb{C} . Let D be a simply connected domain in \mathbb{C} and suppose that $g^n(z) \rightarrow \infty$ as $n \rightarrow \infty$ for all $z \in D$. Then, for any*

compact subset K of D , there exist constants C and n_0 such that

$$|g^n(z')| \leq C|g^n(z)|,$$

for all $z, z' \in K$ and $n \geq n_0$.

We now indicate a proof of our partial result.

Lemma 5.1.3. *Let $f \in \mathcal{F}$ where p is odd. Let U be a wandering domain for f and write $U_n := f^n(U)$ for $n \in \mathbb{N}$. For all $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that, for all U_n with $\text{dist}(U_n, 0) > R$, U_n has to lie inside $Q_{(p+1)/2}^\varepsilon = \left\{ z \exp\left(\frac{i(p+1)\pi}{p}\right) : \text{Re}(z) > 0, |\text{Im}(z)| < q \right\}$, where $Q_{(p+1)/2}^\varepsilon := \left\{ z \exp\left(\frac{i(p+1)\pi}{p}\right) : \text{Re}(z) > 0, |\text{Im}(z)| < q + \varepsilon \right\}$. Note that $Q_{(p+1)/2}^\varepsilon \supset Q_{(p+1)/2}$, defined in (2.0.2).*

Outline proof. Since U is bounded (as $J(f)$ is a spider's web from Theorem 2.0.3), it follows that each U_n is a Fatou component.

From Theorem 1.3.7 and Corollary 1.6.7 it follows that there exists a subsequence (n_k) in \mathbb{N} , for $k > 0$, such that $U_{n_k} \rightarrow \infty$ as $k \rightarrow \infty$.

By Corollary 4.2.4, we have $P(f) \subset \mathbb{R}$. Therefore, by Theorem 1.3.7 and Lemma 4.2.6, we have

$$\text{dist}(U_{n_k}, \mathbb{R}) \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Suppose that, for some arbitrarily large k_0 , we have $U_{n_{k_0}} \cap T_0(\nu) \neq \emptyset$. We can then use the arguments of the proof of Theorem 4.1.1 to get a contradiction. Therefore,

$$\text{dist}(U_{n_k}, \mathbb{R}^-) \rightarrow 0, \text{ as } k \rightarrow \infty.$$

We now use Wiman's theorem [61, p. 274], which implies that for any function of order less than $1/2$ the minimum modulus $m(r)$ has the property that $m(r)/r$ tends to infinity through a sequence of values of r . Applying this result to the function g of Theorem 3.2.2, it follows that $f(x) = x$ for infinitely many points $x < 0$. In fact, the distance between two such points tends to π as $x \rightarrow -\infty$. But these fixed points of f are periodic points of the type described in Lemma 3.3.7, and are consequently endpoints of curves in two Cantor bouquets: the one in $T_{(p+1)/2}(\nu)$ and the one in $T_{(p-1)/2}(\nu)$.

Indeed, by Lemma 3.3.7 and the symmetry property of f , from each of these fixed points there originate two curves that extend to infinity, one lying mostly in $T_{(p+1)/2}(\nu)$ and one lying mostly in $T_{(p-1)/2}(\nu)$. The union of these two curves with respect to each fixed point is thus a curve that partitions

the plane into two “half-planes”. We denote this union of two curves by Γ_m , with m being the relevant value from Lemma 3.3.7. We denote the strip defined between Γ_m and Γ_{m+1} by Δ_m .

Hence, each U_{n_k} , for large enough k , has to lie in Δ_m for some $m \in \mathbb{N}$ that depends on k .

Suppose now that there exists $\varepsilon > 0$ such that $U_{n_k} \cap Q_{(p+1)/2}^\varepsilon \neq \emptyset$ for infinitely many k .

For each of these k , there exists a curve c_k in U_{n_k} , with endpoints in $\partial Q_{(p+1)/2}$ and $\partial Q_{(p+1)/2}^\varepsilon$. Therefore, for the length of each of these curves it holds that

$$\text{length}(c_k) > \varepsilon, \quad \text{for all } k \in \mathbb{N}.$$

Each c_k , then, traverses a distance of at least $\varepsilon/4$ either horizontally or vertically, with respect to the fundamental domains in either $T_{(p+1)/2}(\nu)$ or $T_{(p-1)/2}(\nu)$; that is, parallel to the axis of the domains or orthogonal to the boundary, respectively. Recall that U_{n_k} has to lie in Δ_m , so between Γ_m and Γ_{m+1} for some $m \in \mathbb{N}$.

Suppose that c_k traverses a distance of $\varepsilon/4$ vertically with respect to the fundamental domain it lies in, in either $T_{(p+1)/2}(\nu)$ or $T_{(p-1)/2}(\nu)$. Then $f(c_k)$, for large enough k , has to wind around 0 through an angle of at least $\varepsilon/8$ inside an annulus whose radius increases to infinity as $m \rightarrow \infty$. For sufficiently large m , $f(c_k)$ would then intersect the $J(f)$ Cantor bouquet, which is a contradiction.

Suppose, on the other hand, that c_k traverses a distance of $\varepsilon/4$ horizontally with respect to the direction of fundamental domains, in either $T_{(p+1)/2}(\nu)$ or $T_{(p-1)/2}(\nu)$. Each $f^n(c_k)$, $n \geq 0$, has to lie inside some Δ_{m_n} , where $m_n \in \mathbb{N}$.

Since c_k contains points, z_k and z'_k say, whose distance apart in the direction of the fundamental domains is at least $\varepsilon/4$, then using the fact that $f(z) \sim e^z$ in $T_0(\nu)$ (see proof of Proposition 3.4.3), and also the fact that we have points in the wandering domain arbitrarily close to \mathbb{R} (see (4.3.1)), we deduce that the image curve $f(c_k)$ stretches from $\{z : |z| = |f(z_k)|\}$ to $\{z : |z| = |f(z'_k)|\}$, where $|f(z'_k)| \geq |f(z_k)|e^{\varepsilon/4}$.

Since this image curve, $f(c_k)$, must lie inside some Δ_{m_1} , it must, for large k , traverse a distance comparable to $|f(z_k)| (e^{\varepsilon/4} - 1)$ in the direction of the fundamental domains.

Starting with a sufficiently large k , we thus obtain an image curve that stretches considerably more than $\varepsilon/4$ in the direction of the fundamental domains, and by repeating this process infinitely often we obtain image curves whose length grows at the rate of an iterated exponential, and hence contradict Lemma 5.1.2. \square

5.2 Cantor bouquets in spiders' webs of the larger family

In Chapter 3 we proved the existence of Cantor bouquets in the Julia set spiders' webs of functions in \mathcal{F}_p , $p \geq 3$. We know from Theorem 2.0.3 that the Julia sets of all functions in the larger class \mathcal{E}_p , $p \geq 3$, are also spiders' webs.

Question 5.2.1. Can we prove the existence of Cantor bouquets in the Julia set spiders' webs of functions in \mathcal{E}_p , $p \geq 3$?

Note that the existence of “tails” of hairs can be proven for all functions in \mathcal{E}_p , $p \geq 3$, using Lemma 2.0.1. This is due to the fact that f behaves like a single exponential in each of $T_k(\nu)$, $k = 0, \dots, p-1$, from Lemma 2.0.1. In particular, f behaves like e^z in $T_0(\nu)$, and we can thus apply the techniques we used in Chapter 3 to obtain these tails.

The challenge in this case is that the dynamics in $T(\nu)^c$ depend heavily on the different coefficients a_k in the formula for f (see (2.0.1)) and can thus get very complicated. In particular, locating the endpoints is challenging, as the work done with trapeziums in Section 3.3 cannot be replicated without the symmetry we obtain by setting all coefficients to be the same.

5.3 Long term questions

We devote this section to more general and long term questions.

Question 5.3.1. Does there exist a transcendental entire function whose Julia set spider's web does not contain a Cantor bouquet?

So far, the non-existence of Cantor bouquets in spiders' webs has not been proven. Finding a “special” spider's web as a counter-example would be one way to attack this problem. Another is to even further understand the spider's web structure, as if there is indeed a Cantor bouquet inside every Julia set spider's web, spider's webs are even more elaborate and intricate than previously thought.

Finally, we mention an intriguing question about spider's webs.

Question 5.3.2. Suppose that either the Julia set or escaping set of a transcendental entire function f is a spider's web. Can f have any finite asymptotic values?

We know that when $A_R(f)$ is a spider's web, the function f has no asymptotic values [52, Theorem 1.8]. Can we prove the same for when either $J(f)$ or $I(f)$ is a spider's web?

There are examples of functions where $I(f)$ is a spider's web but $A_R(f)$ is not. Evdoridou's example in [27] of such a function, has no asymptotic values.

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